連続式　ラプラス方程式

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \cdot \nabla \phi = 0$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

連続式

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

海底の境界条件

$$w = -\frac{\partial \phi}{\partial z} = 0 \qquad \text{on } z = -h$$

水面の運動学的境界条件

$$w = -\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \qquad \text{on } \ z = \eta(x, t)$$

$$w = -\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} \qquad \text{on } \ z = \eta(x, t)$$

水面の力学的境界条件

$$-\frac{\partial\phi}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]+g\eta=C(t)$$

$$\text{on}\ \ z=\eta(x,t)$$

$$-\frac{\partial\phi}{\partial t}+\frac{p_\eta}{\rho}+\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]+gz=C(t),$$

on $z=\eta(x,t)$

周期境界

$$\phi(x,t) = \phi(x+L,t)$$

$$\phi(x,t) = \phi(x,t+T)$$

変数分離

$$\phi(x,z,t) = X(x)\cdot Z(z)\cdot T(t)$$

$$\Downarrow$$

$$\phi(x,z,t) = X(x)\cdot Z(z)\cdot \sin\sigma t$$

時間の周期境界

$$\sin \sigma t = \sin \sigma (t + T)$$

$$\sin \sigma t = \sin \sigma t \ \cos \sigma T + \cos \sigma t \ \sin \sigma T$$

↓

$$\cos \sigma T = 1 \qquad \sin \sigma T = 0$$

↓

$$\sigma T = 0, 2\pi$$

↓

$$\sigma = 2\pi / T$$

変数分離

$$\phi(x, z, t) = X(x) \cdot Z(z) \cdot \sin \sigma t$$

$$\frac{d^2 X(x)}{dx^2} \cdot Z(z) \cdot \sin \sigma t + X(x) \cdot \frac{d^2 Z(z)}{dz^2} \cdot \sin \sigma t = 0$$

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = 0$$

変数分離

$$\frac{d^2X(x)/dx^2}{X(x)} = -k^2$$

$$\frac{d^2Z(z)/dz^2}{Z(z)} = +k^2$$

$$\frac{d^2X(x)/dx^2}{X(x)} = -k^2$$

$$X(x) = A\ \cos kx + B\ \sin kx$$

$$\frac{d^2Z(z)/dz^2}{Z(z)} = +k^2$$

$$\mathrm{Z}(x) = Ce^{kz} + De^{-kz}$$

$$\phi(x,z,t)$$
$$= (A\ \cos kx + B\ \sin kx)\left(Ce^{kz} + De^{-kz}\right)\ \sin \sigma t$$

空間の周期境界

$$A\ \cos kx + B\ \sin kx$$
$$= A\ \cos k(x+L) + B\ \sin k(x+L)$$
$$= A(\cos kx\ \cos kL - \sin kx \sin kL)$$
$$+B(\sin kx\ \cos kL + \cos kx\ \sin kL)$$

↓

$$\cos kL = 1 \qquad \sin kL = 0$$

↓

$$kL = 0, 2\pi$$

↓

$$k = 2\pi/L$$

海底の境界条件

$$w = -\frac{\partial \phi}{\partial z} = 0 \qquad \text{on } z = -h$$

$$w = -\frac{\partial \phi}{\partial z}$$

$$= -A \cos kx \left(kCe^{kz} - kDe^{-kz}\right) \sin \sigma t = 0 \qquad \text{on } z = -h$$

⇩

$$-Ak \cos kx \left(Ce^{-kh} - De^{kh}\right) \sin \sigma t = 0$$

⇩

$$C = De^{2kh}$$

海底の境界条件

$$C = De^{2kh}$$

$$\phi = A\cos kx\left(De^{2kh}e^{kz} + De^{-kz}\right)\sin \sigma t$$

$$\phi = ADe^{kh}\cos kx\left(e^{k(h+z)} + e^{-k(h+z)}\right)\sin \sigma t$$

$$\phi = G\ \cos kx \cosh k\,(h+z)\sin \sigma t$$

$$G = 2ADe^{kh}$$

水面の力学的境界条件

$$-\frac{\partial\phi}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]+g\eta=C(t)$$

$$\text{on } z=\eta(x,t)$$

テイラー展開

$$\left(gz-\frac{\partial\phi}{\partial t}+\frac{u^2+w^2}{2}\right)_{z=\eta}=\left(gz-\frac{\partial\phi}{\partial t}+\frac{u^2+w^2}{2}\right)_{z=0}$$

$$+\eta\left[g-\frac{\partial^2\phi}{\partial z\,\partial t}+\frac{1}{2}\frac{\partial}{\partial z}(u^2+w^2)\right]_{z=0}+\cdots=C(t)$$

$$\left(-\frac{\partial \phi}{\partial t} + g\eta\right)_{z=0} = C(t)$$

$$\Rightarrow \quad \eta = \frac{1}{g}\frac{\partial \phi}{\partial t}\bigg|_{z=0} + \frac{C(t)}{g}$$

$$\Rightarrow \quad \eta = \frac{G\sigma}{g}\cos kx \cosh k(h+z)\cos \sigma t\bigg|_{z=0} + \frac{C(t)}{g}$$

$$\Rightarrow \quad = \left[\frac{G\sigma \cosh kh}{g}\right]\cos kx \cos \sigma t + \frac{C(t)}{g}$$

$$\eta = \left[\frac{G\sigma \cosh kh}{g}\right] \cos kx \cos \sigma t + \frac{C(t)}{g}$$

時間平均するとηは0、C（t）＝0

$$\eta = \frac{H}{2} \cos kx \cos \sigma t$$

$$G = \frac{Hg}{2\sigma \cosh kh}$$

$$\phi = \frac{Hg \cosh k(h+z)}{2\sigma \cosh kh} \cos kx \sin \sigma t$$

水面の運動学的境界条件

$$w = -\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \qquad \text{on} \;\; z = \eta(x,t)$$

$$-\frac{H}{2}\,\frac{gk}{\sigma}\,\frac{\sinh k(h+z)}{\cosh kh}\cos kx \sin \sigma t\Big|_{z=0}$$

$$= -\frac{H}{2}\sigma \cos kx \sin \sigma t$$

$$\sigma^2 = gk \tanh kh$$

水面の運動学的境界条件

$$w = -\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \qquad \text{on } \; z = \eta(x, t)$$

$$-\frac{H}{2}\frac{gk}{\sigma}\frac{\sinh k(h+z)}{\cosh kh}\cos kx \sin \sigma t \Big|_{z=0}$$
$$= -\frac{H}{2}\sigma \cos kx \sin \sigma t$$

↓

$$\sigma^2 = gk \tanh kh$$ **分散関係式**

分散関係式

$$\left(\frac{2\pi}{T}\right)^2 = g\frac{2\pi}{L}\tanh kh$$

$$C^2 = \frac{L^2}{T^2} = \frac{g}{k}\tanh kh$$

$$L = \frac{g}{2\pi}T^2\tanh\frac{2\pi h}{L}$$

$$L = L_0\tanh kh$$

$$C = \frac{L_0}{T}\tanh kh$$

Standing Wave

$$\phi = \frac{Hg \cosh k(h+z)}{2\sigma \cosh kh} \cos kx \sin \sigma t$$

$$\phi = \frac{Hg \cosh k(h+z)}{2\sigma \cosh kh} \sin kx \cos \sigma t$$

$$\phi = \frac{H}{2} \frac{g}{\sigma} \frac{\cosh k(h+z)}{\cosh kh} (\cos kx \sin \sigma t - \sin kx \cos \sigma t)$$

$$\phi = -\frac{H}{2} \frac{g}{\sigma} \frac{\cosh k(h+z)}{\cosh kh} \sin(kx - \sigma t)$$

進行波

$$\phi = -\frac{H}{2}\frac{g}{\sigma}\frac{\cosh k(h+z)}{\cosh kh}\sin(kx-\sigma t)$$

$$\eta(x,t) = \frac{1}{g}\frac{\partial \phi}{\partial t}\bigg|_{z=0} = \frac{H}{2}\cos(kx-\sigma t)$$

coshの性質
khが大きい場合

$$\cosh kh = \frac{e^{kh} + e^{-kh}}{2}$$

$$\Rightarrow \quad \cosh kh = \frac{e^{kh}}{2}$$

coshの性質
khが小さい場合

$$\cosh kh = \frac{e^{kh} + e^{-kh}}{2}$$

$$e^{(0+kh)} = e^0 + \left.\frac{de^z}{dz}\right|_{z=0} kh + \left.\frac{d^2e^z}{dz^2}\right|_{z=0} \frac{(kh)^2}{2!} + \cdots$$

$$e^{kh} = 1 + kh + \frac{(kh)^2}{2} + \cdots$$

$$e^{-kh} = 1 - kh + \frac{(kh)^2}{2} + \cdots$$

$$\cosh kh = \frac{1}{2}\left[\left(1 + kh + \frac{(kh)^2}{2}\cdots\right) + \left(1 - kh + \frac{(kh)^2}{2}\cdots\right)\right]$$

$$\cong 1 + \frac{(kh)^2}{2} = 1$$

$\cosh kh$	$\frac{e^{kh}}{2}$	1
$\sinh kh$	$\frac{e^{kh}}{2}$	kh
$\tanh kh$	1	kh

浅い海の場合

$$\sigma^2 = gk \tanh kh = gk^2 h$$

$$\frac{\sigma^2}{k^2} = C^2 = gh$$

$$C = \sqrt{gh}$$

深い海の場合

$$\sigma^2 = gk \tanh kh = gk$$

$$L = L_0$$

$$L_0 = \frac{g}{2\pi} T^2 = 1.56T^2$$

$$C_0 = \frac{g}{2\pi} T = 1.56\,T$$