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by

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A NOTE ON URSELL'S PARAMETER FOR LONG WAVES

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1. Introduction

A critical parameter $\eta_0\lambda^2/h^3$ for long waves was introduced by Ursell in 1953. His course of discussion was developed by the general perturbation series of Lagrangian method and three equations were classified depending on $\eta_0\lambda^2/h^3 \gg 1$, $\eta_0\lambda^2/h^3 \sim 1$, $\eta_0\lambda^2/h^3 \ll 1$. In the present paper, the parameter $\eta_0\lambda^2/h^3$ is directly contained in the equations expanded from Friedrichs's method, which is modified by the different stretching between the depth and the amplitude, and equations are classified by $\eta_0\lambda^2/h^3 < 1$, $\eta_0\lambda^2/h^3 > 1$. By this modification it is implied that $\eta_0\lambda^2/h^3$ is used as a parameter to derive the equations such as $\sigma = (h/\lambda)^2$ used in Friedrichs's expansion method and the other parameter η_0/h is used for the estimation of approximation. It is also shown, in the case $\eta_0\lambda^2/h^3 > 1$, this method gives the same results obtained by Keulegan and Patterson which were derived from the potential theory.

Assuming the motion to be two-dimensional and irrotational, the equations of motions, the continuity equation and the boundary conditions at the free surface $y^* = \eta^* + h^*$ and at the bottom $y^* = 0$ are written as follows in terms of the Euler variables:

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{1}{\rho^*} \frac{\partial p^*}{\partial x^*} \tag{1, 1}$$

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -g^* - \frac{1}{\rho^*} \frac{\partial p^*}{\partial y^*} \tag{1, 2}$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{1, 3}$$

$$\frac{\partial u^*}{\partial y^*} = \frac{\partial v^*}{\partial x^*} \tag{1, 4}$$

$$\left.\begin{aligned} &\frac{\partial \eta^*}{\partial t^*} + u^* \frac{\partial \eta^*}{\partial x^*} - v^* = 0 && (1,\ 5) \\ &p^* = 0 && (1,\ 6) \end{aligned}\right\} \quad \text{at} \quad y^* = \eta^* + h^*$$

$$v^* = 0 \qquad \text{at} \quad y^* = 0 \tag{1, 7}$$

where the asterisk is used for the dimensional variables. Since the depth is constant, the x-axis is taken on the bottom and the y-axis is positive upward. The dimensional variables which are contained in Equations (1,1)-(1,7) are transformed to

$$\left.\begin{array}{ll} u^*=c_0\dfrac{\eta_0^*}{h_0}u & x^*=l_0x \\ v^*=c_0\dfrac{l_0}{h_0}\dfrac{\eta_0^*}{h_0}v & y^*=h_0y \\ p^*=\rho^*g^*h_0p & c_0=\sqrt{g^*h_0} \\ \eta^*=\eta_0^*\eta & h^*=h_0h \\ t^*=\dfrac{l_0}{c_0}t & \end{array}\right\} \quad (1,\ 8)$$

where l_0 denotes a characteristic horizontal length, h_0 a characteristic vertical length, η_0^* a characteristic amplitude, and c_0 a characteristic velocity. Hence, Equations(1,1)-(1,7) are led to non-dimensional equations given by

$$\sigma\left\{\beta\frac{\partial u}{\partial t}+\beta^2u\frac{\partial u}{\partial x}+\frac{\partial p}{\partial x}\right\}+\beta^2v\frac{\partial u}{\partial y}=0 \quad (1,\ 9)$$

$$\sigma\left\{\beta\frac{\partial v}{\partial t}+\beta^2u\frac{\partial v}{\partial x}+1+\frac{\partial p}{\partial y}\right\}+\beta^2v\frac{\partial v}{\partial y}=0 \quad (1,10)$$

$$\sigma\beta\frac{\partial u}{\partial x}+\beta\frac{\partial v}{\partial y}=0 \quad (1,11)$$

$$\beta\frac{\partial u}{\partial y}=\beta\frac{\partial v}{\partial x} \quad (1,12)$$

$$\left.\begin{array}{l} \sigma\left\{\beta\dfrac{\partial \eta}{\partial t}+\beta^2u\dfrac{\partial \eta}{\partial x}\right\}-\beta v=0 \quad (1,13) \\ p=0 \quad (1,14) \end{array}\right\} \text{ at } y=\beta\eta+h$$

$$v=0 \qquad \text{at } y=0 \quad (1,15)$$

where $\sigma=(h_0/l_0)^2$, $\beta=\eta_0^*/h_0$. If $\sigma<\beta$, that is to say, $\eta_0^*l_0^2/h_0^3$ is larger than unity, $\kappa=\sigma/\beta=h_0^3/\eta_0^*l_0^2$ is taken as a parameter of perturbation, and if $\sigma>\beta$, then, $\kappa'=\eta_0^*l_0^2/h_0^3$ is smaller than unity, and κ' is regarded as a parameter of perturbation.

2. The derivation of equations for the case $\eta_0^*l_0^2/h_0^3>1$

Epuations (1,9)-(1,15) are devided by β and are transformed to

$$\kappa\left\{\beta\frac{\partial u}{\partial t}+\beta^2u\frac{\partial u}{\partial t}+\frac{\partial p}{\partial x}\right\}+\beta v\frac{\partial u}{\partial y}=0 \quad (2,\ 1)$$

$$\kappa\left\{\beta\frac{\partial v}{\partial t}+\beta^2u\frac{\partial v}{\partial x}+1+\frac{\partial p}{\partial y}\right\}+\beta v\frac{\partial v}{\partial y}=0 \quad (2,\ 2)$$

$$\kappa\beta\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad (2,\ 3)$$

$$\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x} \qquad (2,\ 4)$$

$$\left.\begin{aligned}&\kappa\left\{\beta\frac{\partial\eta}{\partial t}+\beta^2 u\frac{\partial\eta}{\partial x}\right\}-v=0 \qquad (2,\ 5)\\ &p=0 \qquad (2,\ 6)\end{aligned}\right\}\ \text{at}\quad y=\beta\eta+h$$

$$v=0 \qquad \text{at}\quad y=0 \qquad (2,\ 7)$$

Here, we assume

$$\left.\begin{aligned}u&=u_0+\kappa u_1+\kappa^2 u_2+\cdots\cdots\\ v&=v_0+\kappa v_1+\kappa^2 v_2+\cdots\cdots\\ \eta&=\eta_0+\kappa\eta_1+\kappa^2\eta_2+\cdots\cdots\\ p&=p_0+\kappa p_1+\kappa^2 p_2+\cdots\cdots\end{aligned}\right\} \qquad (2,\ 8)$$

and furthermore, (2,5) and (2,6) must be replaced by Taylor series expanded at $y=\beta\eta_0+h$,

$$\left[\kappa\left\{\beta\frac{\partial\eta}{\partial t}+\beta^2 u\frac{\partial\eta}{\partial x}\right\}-v\right]_{y=\beta\eta_0+h}+\beta(\eta-\eta_0)\frac{\partial}{\partial y}\left[\kappa\left\{\beta\frac{\partial\eta}{\partial t}+\beta^2 u\frac{\partial\eta}{\partial x}\right\}-v\right]_{y=\beta\eta_0+h}$$

$$+\frac{1}{2}\beta^2(\eta-\eta_0)^2\frac{\partial^2}{\partial y^2}\left[\kappa\left\{\beta\frac{\partial\eta}{\partial t}+\beta^2 u\frac{\partial\eta}{\partial x}\right\}-v\right]_{y=\beta\eta_0+h}+\cdots\cdots=0 \qquad (2,\ 9)$$

$$\left[p\right]_{y=\beta\eta_0+h}+\beta(\eta-\eta_0)\left[\frac{\partial p}{\partial y}\right]_{y=\beta\eta_0+h}+\frac{1}{2}\beta^2(\eta-\eta_0)^2\left[\frac{\partial^2 p}{\partial y^2}\right]_{y=\beta\eta_0+h}+\cdots=0 \quad (2,10)$$

From the zeroth order equations of κ, we can obtain

$$v_0\equiv 0 \qquad (2,11)$$

$$u_0=u_0(x,\ t) \qquad (2,12)$$

$$p_0=0 \quad \text{at}\quad y=\beta\eta_0+h \qquad (2,13)$$

2-1. The first order approximation

The equations for the first order of κ are given by

$$\beta\frac{\partial u_0}{\partial t}+\beta^2 u_0\frac{\partial u_0}{\partial x}+\frac{\partial p_0}{\partial x}=0 \qquad (2,14)$$

$$1+\frac{\partial p_0}{\partial y}=0 \qquad (2,15)$$

$$\beta\frac{\partial u_0}{\partial x}+\frac{\partial v_1}{\partial y}=0 \qquad (2,16)$$

$$\frac{\partial u_1}{\partial y}=\frac{\partial v_1}{\partial x} \qquad (2,17)$$

$$\left.\begin{aligned}&\beta\frac{\partial\eta_0}{\partial t}+\beta^2 u_0\frac{\partial\eta_0}{\partial x}-v_1=0 \qquad (2,18)\\ &p_1+\beta\eta_1\frac{\partial p_0}{\partial y}=0 \qquad (2,19)\end{aligned}\right\}\ \text{at}\quad y=\beta\eta_0+h$$

$$v_1=0 \qquad \text{at} \quad y=0 \tag{2, 20}$$

From (2, 13), (2, 15), the hydrostatic pressure can be shown by

$$p_0=\beta\eta_0+h-y \tag{2, 21}$$

, and v_1 is obtained from (2, 16), (2, 20),

$$v_1=-\beta\frac{\partial u_0}{\partial x}y \tag{2, 22}$$

For the determinations of η_0 and u_0, we assume that β is small and we can also expand series with respect to β,

$$\left.\begin{aligned} u_0&=u_0^{(0)}+\beta u_0^{(1)}+\beta^2u_0^{(2)}+\cdots\cdots \\ \eta_0&=\eta_0^{(0)}+\beta\eta_0^{(1)}+\beta^2\eta_0^{(2)}+\cdots\cdots \end{aligned}\right\} \tag{2, 23}$$

Substitutions of (2, 21), (2, 22) and (2, 23) into (2, 14), (2, 18) lead to the first, the second and the third order equations of β as written respectively in the following:

$$\left.\begin{aligned} &\beta\left\{\frac{\partial u_0^{(0)}}{\partial t}+\frac{\partial \eta_0^{(0)}}{\partial x}\right\}=0 \\ &\beta\left\{\frac{\partial \eta_0^{(0)}}{\partial t}+\frac{\partial u_0^{(0)}}{\partial x}h\right\}=0 \end{aligned}\right\} \tag{2, 24}$$

$$\left.\begin{aligned} &\beta^2\left\{\frac{\partial u_0^{(1)}}{\partial t}+u_0^{(0)}\frac{\partial u_0^{(0)}}{\partial x}+\frac{\partial \eta_0^{(1)}}{\partial x}\right\}=0 \\ &\beta^2\left\{\frac{\partial \eta_0^{(1)}}{\partial t}+u_0^{(0)}\frac{\partial \eta_0^{(0)}}{\partial x}+\eta_0^{(0)}\frac{\partial u_0^{(0)}}{\partial x}+\frac{\partial u_0^{(1)}}{\partial x}h\right\}=0 \end{aligned}\right\} \tag{2, 25}$$

$$\left.\begin{aligned} &\beta^3\left\{\frac{\partial u_0^{(2)}}{\partial t}+u_0^{(1)}\frac{\partial u_0^{(0)}}{\partial x}+u_0^{(0)}\frac{\partial u_0^{(1)}}{\partial x}+\frac{\partial \eta_0^{(2)}}{\partial x}\right\}=0 \\ &\beta^3\left\{\frac{\partial \eta_0^{(2)}}{\partial t}+u_0^{(1)}\frac{\partial \eta_0^{(0)}}{\partial x}+u_0^{(0)}\frac{\partial \eta_0^{(1)}}{\partial x}+\frac{\partial u_0^{(2)}}{\partial x}h \right. \\ &\qquad \left. +\eta_0^{(0)}\frac{\partial u_0^{(1)}}{\partial x}+\eta_0^{(1)}\frac{\partial u_0^{(0)}}{\partial x}\right\}=0 \end{aligned}\right\} \tag{2, 26}$$

Equations (2, 24), (2, 25) are transformed to

$$\beta\left\{\frac{\partial^2\eta_0^{(0)}}{\partial t^2}-h\frac{\partial^2\eta_0^{(0)}}{\partial x^2}\right\}=0 \tag{2, 27}$$

$$\beta^2\left\{\frac{\partial^2\eta_0^{(1)}}{\partial t^2}-h\frac{\partial^2\eta_0^{(1)}}{\partial x^2}\right\}=\beta^2h\frac{\partial^2}{\partial x^2}\left\{\frac{3}{2}\frac{(\eta_0^{(0)})^2}{h}\right\} \tag{2, 28}$$

, and from (2, 24), (2, 25), (2, 27), (2, 28), the following equations for a progressive wave traveling in the positive x direction can be obtained:

$$\left.\begin{aligned} \eta_0^{(0)}&=\eta_0^{(0)}(x-\sqrt{h}\,t) \\ u_0^{(0)}&=\frac{1}{\sqrt{h}}\eta_0^{(0)} \\ u_0^{(1)}&=\frac{1}{\sqrt{h}}\eta_0^{(1)} \end{aligned}\right\} \tag{2, 29}$$

Making use of the relations in (2, 29), the third order equations can be given by

$$\beta^3\left\{\frac{\partial^2\eta_0{}^{(2)}}{\partial t^2}-h\frac{\partial^2\eta_0{}^{(2)}}{\partial x^2}\right\}=\beta^3\left\{4\frac{\partial\eta_0{}^{(0)}}{\partial x}\frac{\partial\eta_0{}^{(1)}}{\partial x}+3\eta_0{}^{(1)}\frac{\partial^2\eta_0{}^{(0)}}{\partial x^2}\right.$$
$$+\eta_0{}^{(0)}\frac{\partial^2\eta_0{}^{(1)}}{\partial x^2}-\frac{\partial\eta_0{}^{(0)}}{\partial x}\frac{\partial u_0{}^{(1)}}{\partial t}-\frac{\partial u_0{}^{(0)}}{\partial x}\frac{\partial\eta_0{}^{(1)}}{\partial t}-\eta_0{}^{(0)}\frac{\partial^2 u_0{}^{(1)}}{\partial x\partial t}$$
$$\left.-u_0{}^{(0)}\frac{\partial^2\eta_0{}^{(1)}}{\partial x\partial t}\right\} \qquad (2,30)$$

and

$$u_0{}^{(2)}=\frac{1}{\sqrt{h}}\eta_0{}^{(2)} \qquad (2,31)$$

Putting $\eta-\eta_0{}^{(0)}=\beta\eta_0{}^{(1)}$ and taking $\eta_0{}^{(0)}$ in the right-hand side of (2, 28) approximately as η , (2, 28) yields with an error of $O(\beta^3)$

$$\beta\left\{\frac{\partial^2\eta}{\partial t^2}-h\frac{\partial^2\eta}{\partial x^2}\right\}=\beta^2 h\frac{\partial^2}{\partial x^2}\left\{\frac{3}{2}\frac{\eta^2}{h}\right\} \qquad (2,32)$$

When (2, 32) is returned to the original variables, we can obtain the same equation as shown by Airy,

$$\frac{\partial^2\eta^*}{\partial t^{*2}}-g^*h^*\frac{\partial^2\eta^*}{\partial x^{*2}}=g^*h^*\frac{\partial^2}{\partial x^{*2}}\left\{\frac{3}{2}\frac{\eta^{*2}}{h^*}\right\} \qquad (2,33)$$

2-2. The second order approximation

The second order equations of κ are given by

$$\beta\frac{\partial u_1}{\partial t}+\beta^2 u_0\frac{\partial u_1}{\partial x}+\beta^2 u_1\frac{\partial u_0}{\partial x}+\frac{\partial p_1}{\partial x}+\beta v_1\frac{\partial u_1}{\partial y}=0 \qquad (2,34)$$

$$\beta\frac{\partial v_1}{\partial t}+\beta^2 u_0\frac{\partial v_1}{\partial x}+\frac{\partial p_1}{\partial y}+\beta v_1\frac{\partial v_1}{\partial y}=0 \qquad (2,35)$$

$$\beta\frac{\partial u_1}{\partial x}+\frac{\partial v_2}{\partial y}=0 \qquad (2,36)$$

$$\frac{\partial u_2}{\partial y}=\frac{\partial v_2}{\partial x} \qquad (2,37)$$

$$\left.\begin{aligned}&\beta\frac{\partial\eta_1}{\partial t}+\beta^2 u_0\frac{\partial\eta_1}{\partial x}+\beta^2 u_1\frac{\partial\eta_0}{\partial x}-v_2-\beta\eta_1\frac{\partial v_1}{\partial y}=0 &(2,38)\\ &p_2+\beta\eta_2\frac{\partial p_0}{\partial y}+\beta\eta_1\frac{\partial p_1}{\partial y}=0 &(2,39)\end{aligned}\right\}\ y=\beta\eta_0+h$$

$$v_2=0 \qquad y=0 \qquad (2,40)$$

Firstly, from (2, 17), (2, 22), u_1 can be shown by

$$u_1=u_1(0)-\beta\frac{\partial^2 u_0}{\partial x^2}\frac{y^2}{2} \qquad (2,41)$$

where $u_1(0)=u_1(x,\ 0,\ t)$, then, v_2 is obtained from (2, 36)

$$v_2=-\beta\frac{\partial u_1(0)}{\partial x}y+\beta^2\frac{\partial^3 u_0}{\partial x^3}\frac{y^3}{6} \tag{2,42}$$

Secondly, from (2, 19) and (2, 35), the pressure p_1 is given by

$$p_1=\beta\eta_1+\beta^2\left\{\frac{y^2}{2}-\frac{(\beta\eta_0+h)^2}{2}\right\}$$
$$\times\left\{\frac{\partial^2 u_0}{\partial x\partial t}+\beta u_0\frac{\partial^2 u_0}{\partial x^2}-\beta\frac{\partial u_0}{\partial x}\frac{\partial u_0}{\partial x}\right\} \tag{2,43}$$

In order to coincide u_1 of (2, 34) with that of (2, 38), Equation (2, 34) is applied on $y=\beta\eta_0+h$. Substitutions of (2, 41), (2, 42), (2, 43) for (2, 34), (2, 38) lead to the first and the second order equations of β as shown respectively by

$$\left.\begin{aligned}&\beta\left\{\frac{\partial u_1^{(0)}(0)}{\partial t}+\frac{\partial \eta_1^{(0)}}{\partial x}\right\}=0\\&\beta\left\{\frac{\partial \eta_1^{(0)}}{\partial t}+\frac{\partial u_1^{(0)}(0)}{\partial x}h\right\}=0\end{aligned}\right\} \tag{2,44}$$

$$\left.\begin{aligned}&\beta^2\left\{\frac{\partial u_1^{(1)}(0)}{\partial t}-\frac{\partial^3 u_0^{(0)}}{\partial x^2\partial t}\frac{h^2}{2}+\frac{\partial \eta_1^{(1)}}{\partial x}\right\}=0\\&\beta^2\left\{\frac{\partial \eta_1^{(1)}}{\partial t}+\frac{\partial u_1^{(1)}(0)}{\partial x}h-\frac{\partial^3 u_0^{(0)}}{\partial x^3}\frac{h^3}{6}\right\}=0\end{aligned}\right\} \tag{2,45}$$

where the following relations are taken into consideration:

$$\left.\begin{aligned}&u_1^{(0)}=u_1^{(0)}(0)\\&u_1^{(1)}=u_1^{(1)}(0)-\frac{\partial^2 u_0^{(0)}}{\partial x^2}\frac{h^2}{2}\\&u_1^{(2)}=u_1^{(2)}(0)-\frac{\partial^2 u_0^{(1)}}{\partial x^2}\frac{h^2}{2}-\eta_0^{(0)}\frac{\partial^2 u_0^{(0)}}{\partial x^2}h\\&v_2^{(0)}=0\\&v_2^{(1)}=-\frac{\partial u_1^{(0)}(0)}{\partial x}h\\&v_2^{(2)}=-\frac{\partial u_1^{(1)}(0)}{\partial x}h-\frac{\partial u_1^{(0)}(0)}{\partial x}\eta_0^{(0)}+\frac{\partial^3 u_0^{(0)}}{\partial x^3}\frac{h^3}{6}\\&v_2^{(3)}=-\frac{\partial u_1^{(2)}(0)}{\partial x}h-\frac{\partial u_1^{(1)}(0)}{\partial x}\eta_0^{(0)}-\frac{\partial u_1^{(0)}}{\partial x}\eta_0^{(1)}\\&\qquad+\frac{\partial^3 u_0^{(1)}}{\partial x^3}\frac{h^3}{6}+\frac{\partial^3 u_0^{(0)}}{\partial x^3}\eta_0^{(0)}\frac{h^2}{2}\end{aligned}\right\}\text{ at } y=\beta\eta_0+h \tag{2,46}$$

Since the motion expressed by (2, 44) is the same as given in (2, 24), the first order wave motion with respect to β is represented by (2, 24) of the first order approximation of κ, which is determined by a proper boundary condition, then, $\eta_1^{(0)}$ and $u_1^{(0)}$ can be made equal to zero. Accordingly, (2, 45) is transformed to

$$\beta^2\left\{\frac{\partial^2\eta_1^{(1)}}{\partial t^2}-h\frac{\partial^2\eta_1^{(1)}}{\partial x^2}\right\}=\beta^2\frac{h^3}{3}\frac{\partial^4\eta_0^{(0)}}{\partial x^4} \tag{2, 47}$$

and

$$u_1^{(1)}(0)=\frac{1}{\sqrt{h}}\eta_1^{(1)} \tag{2, 48}$$

Let η be $\eta-\eta_0^{(0)}=\beta\eta_0^{(1)}+\kappa\beta\eta_1^{(1)}$, we can obtain from (2, 27), (2, 32), (2, 47)

$$\beta\left\{\frac{\partial^2\eta}{\partial t^2}-h\frac{\partial^2\eta}{\partial x^2}\right\}=\beta^2 h\frac{\partial^2}{\partial x^2}\left\{\frac{3}{2}\frac{\eta^2}{h}+\kappa\frac{h^2}{3}\frac{\partial^2\eta}{\partial x^2}\right\} \tag{2, 49}$$

where the terms in the right-hand side are replaced by η, and the error is of order $O(\beta^3)$.

On the other hand, if $\eta_0^{(0)}$ is equal to η which is a solution of (2, 49) from the beginning, $\eta_0^{(1)}$ and $\eta_1^{(1)}$ can be neglected. However, the phase velocity of η in (2, 49) is approximately given by

$$c=\sqrt{h}\left\{1+\beta\frac{3}{4}\frac{\eta}{h}+\beta\kappa\frac{h^2}{6\eta}\frac{\partial^2\eta}{\partial x^2}\right\} \tag{2, 50}$$

, therefore, the error contained in (2, 49) is generally of order $O(\beta^3)$ due to the transformation given by

$$\frac{\partial}{\partial t}=-c\frac{\partial}{\partial x} \tag{2, 51}$$

Returning to the original equation, (2, 49) is transformed to

$$\frac{\partial^2\eta^*}{\partial t^{*2}}-g^*h^*\frac{\partial^2\eta^*}{\partial x^{*2}}=g^*h^*\frac{\partial^2}{\partial x^{*2}}\left\{\frac{3}{2}\frac{\eta^{*2}}{h^*}+\frac{h^{*2}}{3}\frac{\partial^2\eta^*}{\partial x^{*2}}\right\} \tag{2, 52}$$

As for the velocity on the bottom, putting

$$u(0)=u_0^{(0)}+\beta u_0^{(1)}+\kappa\beta u_1^{(1)}(0)$$
$$\eta=\eta_0^{(0)}+\beta\eta_0^{(1)}+\kappa\beta\eta_1^{(1)}$$

we can obtain from the lower equations of (2, 24), (2, 25), (2, 45)

$$\beta\frac{\partial\eta}{\partial t}+\beta\frac{\partial u(0)}{\partial x}h+\beta^2\frac{\partial}{\partial x}\left\{u_0^{(0)}\eta_0^{(0)}\right\}-\kappa\beta^2\frac{\partial^3 u_0^{(0)}}{\partial x^3}\frac{h^3}{6}=0 \tag{2, 53}$$

Considering (2, 50), (2, 51), and integrating (2, 53) with x, $u(0)$ is given by

$$\beta u(0)=\frac{1}{\sqrt{h}}\left\{\beta\eta-\beta^2\frac{1}{4}\frac{\eta^2}{h}+\kappa\beta^2\frac{h^2}{3}\frac{\partial^2\eta}{\partial x^2}\right\} \tag{2, 54}$$

where $\eta_0^{(0)}$ is replaced by η in the right-hand side and the error is also of order $O(\beta^3)$. From (2, 41), the horizontal velocity at an arbitrary point is given by

$$\beta u=\frac{1}{\sqrt{h}}\left\{\beta\eta-\beta^2\frac{1}{4}\frac{\eta^2}{h}+\kappa\beta^2\left(\frac{h^2}{3}-\frac{y^2}{2}\right)\frac{\partial^2\eta}{\partial x^2}\right\} \tag{2, 55}$$

On the other hand, there is no vertical velocity of the zeroth order of β, then, the vertical velocity is obtained, to the third order of β, from (2,22), (2,42),

$$\beta v=\beta\kappa v_1+\beta\kappa^2 v_2$$

$$=-\kappa\beta^2\frac{\partial u(0)}{\partial x}y+\kappa^2\beta^3\frac{\partial^3 u_0^{(0)}}{\partial x^3}\cdot\frac{y^3}{6}$$

$$=-\kappa\beta^2\frac{y}{\sqrt{h}}\left\{\frac{\partial\eta}{\partial x}-\beta\frac{1}{2}\frac{\eta}{h}\frac{\partial\eta}{\partial x}+\kappa\beta\left(\frac{h^2}{3}-\frac{y^2}{6}\right)\frac{\partial^3\eta}{\partial x^3}\right\} \quad (2,56)$$

When (2,55), (2,56) are returned to the original variables, the following expressions can be given:

$$u^*=\sqrt{\frac{g^*}{h^*}}\left\{\eta^*-\frac{\eta^{*2}}{4h^*}+\left(\frac{h^{*2}}{3}-\frac{y^{*2}}{2}\right)\frac{\partial^2\eta^*}{\partial x^{*2}}\right\} \quad (2,57)$$

$$v^*=-\sqrt{\frac{g^*}{h^*}}y^*\left\{\left(1-\frac{\eta^*}{2h^*}\right)\frac{\partial\eta^*}{\partial x^*}+\left(\frac{h^{*2}}{3}-\frac{y^{*2}}{6}\right)\frac{\partial^3\eta^*}{\partial x^{*3}}\right\} \quad (2,58)$$

Equations (2,52), (2,57) and (2,58) are the same as shown by Keulegan and Patterson.

The third order equations of β are given by

$$\left.\begin{aligned}
&\beta^3\left\{\frac{\partial u_1^{(2)}(0)}{\partial t}-\frac{\partial^3 u_0^{(0)}}{\partial t\partial x^2}\frac{h^2}{2}-\eta_0^{(0)}\frac{\partial^3 u_0^{(0)}}{\partial x^2\partial t}h-\frac{\partial\eta_0^{(0)}}{\partial t}\frac{\partial^2 u_0^{(0)}}{\partial x^2}h\right.\\
&+u_0^{(0)}\frac{\partial u_1^{(1)}(0)}{\partial x}-u_0^{(0)}\frac{\partial^3 u_0^{(0)}}{\partial x^3}\frac{h^2}{2}+u_1^{(1)}(0)\frac{\partial u_0^{(0)}}{\partial x}\\
&\left.-\frac{\partial u_0^{(0)}}{\partial x}\frac{\partial^2 u_0^{(0)}}{\partial x^2}\frac{h^2}{2}+\frac{\partial\eta_1^{(2)}}{\partial x}+\frac{\partial u_0^{(0)}}{\partial x}\frac{\partial^2 u_0^{(0)}}{\partial x^2}h^2\right\}=0\\
&\beta^3\left\{\frac{\partial\eta_1^{(2)}}{\partial t}+u_0^{(0)}\frac{\partial\eta_1^{(1)}}{\partial x}+u_1^{(1)}(0)\frac{\partial\eta_0^{(0)}}{\partial x}-\frac{\partial\eta_0^{(0)}}{\partial x}\frac{\partial^2 u_0^{(0)}}{\partial x^2}\frac{h^2}{2}\right.\\
&+\eta_0^{(0)}\frac{\partial u_1^{(1)}(0)}{\partial x}+\frac{\partial u_1^{(2)}(0)}{\partial x}h-\frac{\partial^3 u_0^{(1)}}{\partial x^3}\frac{h^3}{6}-\frac{\partial^3 u_0^{(0)}}{\partial x^3}\eta_0^{(0)}\frac{h^2}{2}\\
&\left.+\eta_1^{(1)}\frac{\partial u_0^{(0)}}{\partial x}\right\}=0
\end{aligned}\right\} \quad (2,59)$$

, therefore, (2,59) is transformed to

$$\begin{aligned}
\beta^3\left\{\frac{\partial^3\eta_1^{(2)}}{\partial t^2}-h\frac{\partial^2\eta_1^{(2)}}{\partial x^2}\right\}=\beta^3&\left\{4\frac{\partial\eta_0^{(0)}}{\partial x}\frac{\partial\eta_1^{(1)}}{\partial x}+3\eta_1^{(1)}\frac{\partial^2\eta_0^{(0)}}{\partial x^2}+\eta_0^{(0)}\frac{\partial^2\eta_1^{(1)}}{\partial x^2}\right.\\
&-\frac{\partial\eta_0^{(0)}}{\partial x}\frac{\partial u_1^{(1)}(0)}{\partial t}-\frac{\partial u_0^{(0)}}{\partial x}\frac{\partial\eta_1^{(1)}}{\partial t}-\eta_0^{(0)}\frac{\partial^2 u_1^{(1)}(0)}{\partial x\partial t}\\
&-u_0^{(0)}\frac{\partial^2\eta_1^{(1)}}{\partial x\partial t}-\frac{\partial^4 u_0^{(1)}}{\partial x^3\partial t}\frac{h^3}{3}+\frac{\partial^2\eta_0^{(0)}}{\partial x^2}\cdot\frac{\partial^2\eta_0^{(0)}}{\partial x^2}h^2\\
&\left.+\frac{\partial\eta_0^{(0)}}{\partial x}\cdot\frac{\partial^3\eta_0^{(0)}}{\partial x^3}h^2\right\}
\end{aligned} \quad (2,60)$$

2-3. The third order approximation

The third order equations of κ are given by

$$\beta\frac{\partial u_2}{\partial t}+\beta^2 u_0\frac{\partial u_2}{\partial x}+\beta^2 u_1\frac{\partial u_1}{\partial x}+\beta^2 u_2\frac{\partial u_0}{\partial x}+\frac{\partial p_2}{\partial x}+\beta v_1\frac{\partial u_2}{\partial y}$$

$$+\beta v_2\frac{\partial u_1}{\partial y}=0 \qquad (2,61)$$

$$\beta\frac{\partial v_2}{\partial t}+\beta^2 u_0\frac{\partial v_2}{\partial x}+\beta^2 u_1\frac{\partial v_1}{\partial x}+\frac{\partial p_2}{\partial y}+\beta v_1\frac{\partial v_2}{\partial y}+\beta v_2\frac{\partial v_1}{\partial y}=0 \qquad (2,62)$$

$$\beta\frac{\partial u_2}{\partial x}+\frac{\partial v_3}{\partial y}=0 \qquad (2,63)$$

$$\frac{\partial u_3}{\partial y}=\frac{\partial v_3}{\partial x} \qquad (2,64)$$

$$\left.\begin{aligned}&\beta\frac{\partial \eta_2}{\partial t}+\beta^2 u_0\frac{\partial \eta_2}{\partial x}+\beta^2 u_1\frac{\partial \eta_1}{\partial x}+\beta^2 u_2\frac{\partial \eta_0}{\partial x}-v_3+\beta^3\eta_1\frac{\partial u_1}{\partial y}\frac{\partial \eta_0}{\partial x} \qquad (2,65)\\&-\beta\eta_1\frac{\partial v_2}{\partial y}-\beta\eta_2\frac{\partial v_1}{\partial y}=0\\&p_3+\beta\eta_3\frac{\partial p_0}{\partial y}+\beta\eta_2\frac{\partial p_1}{\partial y}+\beta\eta_1\frac{\partial p_2}{\partial y}+\frac{1}{2}\beta^2\eta_1{}^2\frac{\partial^2 p_1}{\partial y^2}=0 \qquad (2,66)\end{aligned}\right\}\text{ at } y=\beta\eta_0+\mathrm{h}$$

$$v_3=0 \qquad \text{at} \quad y=0 \qquad (2,67)$$

From (2,39), (2,62), p_2 can be obtained to the third order of β,

$$p_2=\beta\eta_2-\beta^3\eta_1\frac{\partial^2 u_0}{\partial x\partial t}h+\left\{\beta^2\frac{\partial^2 u_1(0)}{\partial x\partial t}+\beta^3 u_0\frac{\partial^2 u_1(0)}{\partial x^2}\right.$$

$$\left.+\beta^3 u_1(0)\frac{\partial^2 u_0}{\partial x^2}-\beta^3\cdot 2\frac{\partial u_0}{\partial x}\frac{\partial u_1(0)}{\partial x}\right\}\left\{\frac{y^2}{2}-\frac{(\beta\eta_0+h)^2}{2}\right\} \qquad (2,68)$$

, since $\eta_1{}^{(0)}$ and $u_1{}^{(0)}(0)$ are neglected, p_2 becomes

$$p_2=\beta\eta_2+\beta^3\frac{\partial^2 u_1{}^{(1)}(0)}{\partial x\partial t}\left\{\frac{y^2}{2}-\frac{(\beta\eta_0+h)^2}{2}\right\} \qquad (2,69)$$

From (2,37), (2,47), u_2 can be given by

$$u_2=u_2(0)-\beta\frac{\partial^2 u_1(0)}{\partial x^2}\cdot\frac{y^2}{2}+\beta^2\frac{\partial^4 u_0}{\partial x^4}\frac{y^4}{24} \qquad (2,70)$$

, and v_3 is also obtained from (2,63), (2,70),

$$v_3=-\beta\frac{\partial u_2(0)}{\partial x}y+\beta^2\frac{\partial^2 u_1(0)}{\partial x^2}\frac{y^3}{6}-\beta^3\frac{\partial^5 u_0}{\partial x^5}\frac{y^5}{120} \qquad (2,71)$$

In the same way as the second order approximation, applying (2,61) on $y=\beta\eta_0+h$, the first and the second order equations of β can be given respectively by

$$\left.\begin{aligned}&\beta\left\{\frac{\partial u_2{}^{(0)}(0)}{\partial t}+\frac{\partial \eta_2{}^{(0)}}{\partial x}\right\}=0\\&\beta\left\{\frac{\partial \eta_2{}^{(0)}}{\partial t}+\frac{\partial u_2{}^{(0)}(0)}{\partial x}h\right\}=0\end{aligned}\right\} \qquad (2,72)$$

$$\beta^2\left\{\frac{\partial u_2{}^{(1)}(0)}{\partial t}+\frac{\partial \eta_2{}^{(1)}}{\partial x}\right\}=0$$

$$\beta^2\left\{\frac{\partial \eta_2^{(1)}}{\partial t}+\frac{\partial u_2^{(1)}(0)}{\partial x}h\right\}=0 \qquad (2,73)$$

Since (2, 72) and (2, 73) are the same equations as (2, 24), $\eta_2^{(0)}$, $u_2^{(0)}(0)$, $\eta_2^{(1)}$, $u_2^{(1)}(0)$ can be neglected. Consequently, the equation to the second order of β is independent of κ higher than the second order and the rapidity of convergence depends on β. The third order equations of β are given by,

$$\left.\begin{aligned}&\beta^3\left\{\frac{\partial u_2^{(2)}(0)}{\partial t}-\frac{\partial^3 u_1^{(1)}(0)}{\partial x^2\partial t}\frac{h^2}{2}+\frac{\partial_5 u_0^{(0)}}{\partial x^4\partial t}\frac{h^4}{24}+\frac{\partial \eta_2^{(2)}}{\partial x}\right\}=0\\&\beta^3\left\{\frac{\partial \eta_2^{(2)}}{\partial t}+\frac{\partial u_2^{(2)}(0)}{\partial x}h-\frac{\partial^3 u_1^{(1)}(0)}{\partial x^3}\frac{h^3}{6}+\frac{\partial^5 u_0^{(0)}}{\partial x^5}\frac{h^5}{120}\right\}=0\end{aligned}\right\} \qquad (2,74)$$

then, (2, 74) is transformed to

$$\beta^3\left\{\frac{\partial^2 \eta_2^{(2)}}{\partial t^2}-h\frac{\partial^2 \eta_2^{(2)}}{\partial x^2}\right\}=\beta^3\left\{\frac{h^5}{30}\frac{\partial^6 u_0^{(0)}}{\partial x^5\partial t}-\frac{\partial^4 u_1^{(1)}(0)}{\partial x^3\partial t}\frac{h^3}{3}\right\} \qquad (2,75)$$

Accordingly, let $\eta_0^{(0)}$ be η_s and the solution of (2, 49), we can obtain from (2, 30), (2, 60), (2, 75)

$$\begin{aligned}\beta\left\{\frac{\partial^2 \eta}{\partial t^2}-h\frac{\partial^2 \eta}{\partial x^2}\right\}=\beta^2 h\frac{\partial^2}{\partial x^2}\left\{\frac{3}{2}\frac{\eta_s^2}{h}+\kappa\frac{h^2}{3}\frac{\partial^2 \eta_s}{\partial x^2}\right.\\\left.+\beta\kappa\left(\frac{\partial \eta_s}{\partial x}\right)^2\frac{h}{2}-\beta\kappa^2\frac{h^4}{30}\frac{\partial^4 \eta_s}{\partial x^4}\right\}\end{aligned} \qquad (2,76)$$

where $\eta-\eta_s=\beta^2\eta_0^{(2)}+\kappa\beta^2\eta_1^{(2)}+\kappa^2\beta^2\eta_2^{(2)}$.

Furthermore, when η_s is replaced by η in the right-hand side, (2, 76) can be transformed to

$$\begin{aligned}\beta\left\{\frac{\partial^2 \eta}{\partial t^2}-h\frac{\partial^2 \eta}{\partial x^2}\right\}=\beta^2 h\frac{\partial^2}{\partial x^2}\left\{\frac{3}{2}\frac{\eta^2}{h}+\kappa\frac{h^2}{3}\frac{\partial^2 \eta}{\partial x^2}\right.\\\left.+\beta\kappa\frac{h}{2}\left(\frac{\partial \eta}{\partial x}\right)^2-\beta\kappa^2\frac{h^4}{30}\frac{\partial^4 \eta}{\partial x^4}\right\}\end{aligned} \qquad (2,77)$$

However, generally, (2, 49) has an error of $O(\beta^3)$, only for the case where, in Equation (2, 50),

$$\frac{3}{4}\frac{\eta}{h}+\kappa\frac{h^2}{6\eta}\frac{\partial^2 \eta}{\partial x^2}\sim\beta \qquad (2,78)$$

, namely, the phase velocity is very near to $\sqrt{h}$, Equation (2, 77) is being established on the error of $O(\beta^4)$.

Returning to the original variables, (2, 77) becomes

$$\begin{aligned}\frac{\partial^2 \eta^*}{\partial t^{*2}}-g^*h^*\frac{\partial^2 \eta^*}{\partial x^{*2}}=g^*h^*\frac{\partial^2}{\partial x^{*2}}\left\{\frac{3}{2}\frac{\eta^{*2}}{h^*}+\frac{h^{*2}}{3}\frac{\partial^2 \eta^*}{\partial x^{*2}}\right.\\\left.+\frac{h^*}{2}\left(\frac{\partial \eta^*}{\partial x^*}\right)^2-\frac{h^{*4}}{30}\frac{\partial^4 \eta^*}{\partial x^{*4}}\right\}\end{aligned} \qquad (2,79)$$

3. The derivation of equations for the case $\eta_0{}^*l_0{}^2/h_0{}^3<1$

In this case, Equations (1, 9)-(1, 15) are devided by σ and are transformed to

$$\beta\frac{\partial u}{\partial t}+\beta^2 u\frac{\partial u}{\partial x}+\frac{\partial p}{\partial x}+\kappa'\beta v\frac{\partial u}{\partial y}=0 \qquad (3,\ 1)$$

$$\beta\frac{\partial v}{\partial t}+\beta^2 u\frac{\partial v}{\partial x}+1+\frac{\partial p}{\partial y}+\kappa'\beta v\frac{\partial v}{\partial y}=0 \qquad (3,\ 2)$$

$$\beta\frac{\partial u}{\partial x}+\kappa'\frac{\partial v}{\partial y}=0 \qquad (3,\ 3)$$

$$\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x} \qquad (3,\ 4)$$

$$\left.\begin{aligned}&\beta\frac{\partial \eta}{\partial t}+\beta^2 u\frac{\partial \eta}{\partial x}-\kappa' v=0 &(3,\ 5)\\ &p=0 &(3,\ 6)\end{aligned}\right\}\quad \text{at}\quad y=\beta\eta+h$$

$$v=0 \qquad \text{at} \quad y=0 \qquad (3,\ 7)$$

where $\kappa'=\dfrac{n_0{}^*l_0{}^2}{h_0{}^3}$

Equations (3, 5), (3, 6) are expanded at $y=\beta\eta_0+h$ as shown by

$$\left[\beta\frac{\partial\eta}{\partial t}+\beta^2u\frac{\partial\eta}{\partial x}-\kappa'v\right]_{y=\beta\eta_0+h}+\beta(\eta-\eta_0)\frac{\partial}{\partial y}\left[\beta\frac{\partial\eta}{\partial t}+\beta^2u\frac{\partial\eta}{\partial x}-\kappa'v\right]_{y=\beta\eta_0+h}$$

$$+\frac{1}{2}\beta^2(\eta-\eta_0)^2\frac{\partial^2}{\partial y^2}\left[\beta\frac{\partial\eta}{\partial t}+\beta^2u\frac{\partial\eta}{\partial x}-\kappa'v\right]_{y=\beta\eta_0+h}+\cdots=0 \qquad (3,\ 8)$$

$$\left[p\right]_{y=\beta\eta_0+h}+\beta(\eta-\eta_0)\left[\frac{\partial p}{\partial y}\right]_{y=\beta\eta_0+h}+\frac{1}{2}\beta^2(\eta-\eta_0)^2\left[\frac{\partial^2 p}{\partial y^2}\right]_{y=\beta\eta_0+h}$$

$$+\cdots=0 \qquad (3,\ 9)$$

As shown in (3, 3), the vertical velocity is determined by the higher order horizontal velocity, therefore, a different way of discussion is developed.

On an arbitrary n-th order of κ', v_{n-1} can be given by

$$v_{n-1}=-\beta\int_0^y\frac{\partial u_n}{\partial x}dy \qquad (3,10)$$

, and u_{n-1} is obtained from (3, 4),

$$u_{n-1}=u_{n-1}(0)-\beta\int_0^y\int_0^y\frac{\partial^2 u_n}{\partial x^2}dy\,dy \qquad (3,11)$$

In the same way, the following expressions can be obtained:

$$v_{n-m}=(-\beta)^m\underbrace{\int_0^y\cdots\int_0^y}_{(2m-1)}\frac{\partial^{2m-1}u_n}{\partial x^{2m-1}}dy\cdots dy$$

$$+(-\beta)^{m-1}\underbrace{\int_0^y\cdots\int_0^y}_{(2m-3)}\frac{\partial^{2m-3}u_{n-1}(0)}{\partial x^{2m-3}}dy\cdots dy$$

$$+(-\beta)^{m-2}\underbrace{\int_0^y\cdots\int_0^y}_{(2m-5)}\frac{\partial^{2m-5}u_{n-2}(0)}{\partial x^{2m-5}}dy\cdots dy$$

$$+\cdots\cdots\cdots\cdots$$

$$+(-\beta)\int_0^y\frac{\partial u_{n-m+1}(0)}{\partial x}dy$$

$$=(-\beta)^m\underbrace{\int_0^y\cdots\int_0^y}_{(2m-1)}\frac{\partial^{2m-1}u_n}{\partial x^{2m-1}}dy\cdots dy$$

$$+(-\beta)^{m-1}\frac{\partial^{2m-3}u_{n-1}(0)}{\partial x^{2m-3}}\cdot\frac{y^{2m-3}}{(2m-3)!}$$

$$+\cdots\cdots\cdots\cdots$$

$$+(-\beta)\frac{\partial u_{n-m+1}(0)}{\partial x}y \qquad (3,12)$$

and

$$u_{n-m}=u_{n-m}(0)+(-\beta)^m\underbrace{\int_0^y\cdots\int_0^y}_{(2m)}\frac{\partial^{2m}u_n}{\partial x^{2m}}dy\cdots dy$$

$$+(-\beta)^{m-1}\underbrace{\int_0^y\int_0^y\cdots\int_0^y}_{(2m-2)}\frac{\partial^{2m-2}u_n(0)}{\partial x^{2m-2}}dy\cdots dy$$

$$+\cdots\cdots$$

$$+(-\beta)\int_0^y\int_0^y\frac{\partial^2 u_{n-m+1}(0)}{\partial x^2}dy\,dy$$

$$=u_{n-m}(0)+(-\beta)^m\underbrace{\int_0^y\cdots\int_0^y}_{(2m)}\frac{\partial^{2m}u_n}{\partial x^{2m}}dy\cdots dy$$

$$+(-\beta)^{m-1}\frac{\partial^{2m-2}u_{n-1}(0)}{\partial x^{2m-2}}\frac{y^{2m-2}}{(2m-2)!}$$

$$+\cdots\cdots$$

$$+(-\beta)\frac{\partial^2 u_{n-m+1}(0)}{\partial x^2}\frac{y^2}{2} \qquad (3,13)$$

Substituting (3,12), (3,13) into (3,2), the pressure of the n-th order can be given, to the second order of β, by

$$\frac{\partial p_n}{\partial y}=-\beta\frac{\partial v_n}{\partial t}=\beta^2\int_0^y\frac{\partial^2 u_{n+1}}{\partial x\partial t}dy \qquad (3,14)$$

, then,

$$p_n = p_n(0) + \beta^2 \int_0^y \int_0^y \frac{\partial^2 u_{n+1}}{\partial x \partial t} dy\, dy \tag{3,15}$$

and the pressure to the second order of β at $y=\beta\eta_0+h$ is obtained from (3,9), (3,14),

$$p_n + \beta\eta_n \frac{\partial p_0}{\partial y} = 0 \tag{3,16}$$

where

$$\frac{\partial p_0}{\partial y} = -1 - \beta\frac{\partial v_0}{\partial t} - \beta^2 u_0 \frac{\partial v_0}{\partial x}$$

$$= -1 + \beta^2 \int_0^y \frac{\partial^2 u_1}{\partial x \partial t} dy - \beta^2 u_0 \frac{\partial u_0}{\partial y}$$

Consequently, we can obtain

$$p_n = \beta\eta_n \qquad \text{at} \quad y=\beta\eta_0+h \tag{3,17}$$

, therefore, Equation (3,15) can be transformed to

$$p_n = \beta\eta_n - \beta^2 \int_0^{\beta\eta_0+h} \int_0^y \frac{\partial^2 u_{n+1}}{\partial x \partial t} dy\, dy$$

$$+ \beta^2 \int_0^y \int_0^y \frac{\partial^2 u_{n+1}}{\partial x \partial t} dy\, dy \tag{3,18}$$

Noting (3,1), (3,8), (3,10), (3,18), the first order equations of β can be given by

$$\left.\begin{aligned} &\beta\left\{\frac{\partial u_n^{(0)}}{\partial t} + \frac{\partial \eta_n^{(0)}}{\partial x}\right\} = 0 \\ &\beta\left\{\frac{\partial \eta_n^{(0)}}{\partial t} + \int_0^{\beta\eta_0+h} \frac{\partial u_n^{(0)}}{\partial x} dy\right\} = 0 \end{aligned}\right\} \tag{3,19}$$

As for $\beta\eta_0$, we can evaluate from the zeroth order equations of κ', that is,

$$\beta\frac{\partial u_0}{\partial t} + \beta^2 u_0 \frac{\partial u_0}{\partial x} + \frac{\partial p_0}{\partial x} = 0 \tag{3,20}$$

$$\beta\frac{\partial v_0}{\partial t} + \beta^2 u_0 \frac{\partial v_0}{\partial x} + 1 + \frac{\partial p_0}{\partial y} = 0 \tag{3,21}$$

$$\beta\frac{\partial u_0}{\partial x} = 0 \tag{3,22}$$

$$\frac{\partial u_0}{\partial y} = \frac{\partial v_0}{\partial x} \tag{3,23}$$

$$\beta\frac{\partial \eta_0}{\partial t} + \beta^2 u_0 \frac{\partial \eta_0}{\partial x} = 0 \tag{3,24}$$

$$p_0 = 0 \tag{3,25}$$

(3,24) and (3,25) at $y=\beta\eta_0+h$

$$v_0 = 0 \qquad \text{at} \quad y=0 \tag{3,26}$$

From (3,24), we can obtain

$$\frac{\partial \eta_0^{(0)}}{\partial t} = 0$$

, and from (3,18), (3,20), (3,22), we can also obtain

$$\beta\frac{\partial u_0}{\partial t}+\beta\frac{\partial\eta_0}{\partial x}-\beta^2\int_0^{\beta\eta_0+h}\int_0^y\frac{\partial^3 u_1}{\partial t\partial x^2}dy\,dy+\beta^2\int_0^y\int_0^y\frac{\partial^3 u_1}{\partial t\partial x^2}dy\,dy$$

$$-\beta^3\frac{\partial\eta_0}{\partial x}\int_0^{\beta\eta_0+h}\frac{\partial^3 u_1}{\partial t\partial x^2}dy=0 \qquad (3,27)$$

Taking consideration of the first order of β in (3, 27), $u_0^{(0)}$ is not a function of x, and $\eta_0^{(0)}$ is not a function of t and y, therefore, we can treat $\eta_0^{(0)}$ and $u_0^{(0)}$ as

$$\eta_0^{(0)}=\text{Const.}$$

$$u_0^{(0)}=\text{Const.}$$

Putting $\eta_0^{(0)}=0$, (3, 19) can be transformed to

$$\beta\left\{\frac{\partial^2\eta_n^{(0)}}{\partial t^2}-h\frac{\partial^2\eta_n^{(0)}}{\partial x^2}\right\}=0 \qquad (3,28)$$

If we take a sufficiently large n and neglect $\eta_{k'}^{(0)}$ and $u_{k'}^{(0)}$ except for $\eta_n^{(0)}$ and $u_n^{(0)}$, the second order equations of β can be obtained from (3, 1), (3, 8), (3, 18),

$$\beta^2\left\{\frac{\partial u_{n-1}^{(1)}}{\partial t}+\frac{\partial\eta_{n-1}^{(1)}}{\partial x}-\int_0^h\int_0^y\frac{\partial^3 u_n^{(0)}}{\partial t\partial x^2}dy\,dy+\int_0^y\int_0^y\frac{\partial^3 u_n^{(0)}}{\partial t\partial x^2}dy\,dy\right\}=0$$

$$\beta^2\left\{\frac{\partial\eta_{n-1}^{(1)}}{\partial t}+\int_0^h\frac{\partial u_{n-1}^{(1)}}{\partial x}dy\right\}=0 \qquad (3,29)$$

, therefore, (3, 29) is transformed to

$$\beta^2\left\{\frac{\partial^2\eta_{n-1}^{(1)}}{\partial t^2}-h\frac{\partial^2\eta_{n-1}^{(1)}}{\partial x^2}\right\}=\beta^2\left\{\int_0^h\int_0^y\int_0^y\frac{\partial^4 u_n^{(0)}}{\partial t\partial x^3}dy\,dy\,dy\right.$$

$$\left.-\int_0^h\int_0^h\int_0^y\frac{\partial^4 u_n^{(0)}}{\partial t\partial x^3}dy\,dy\,dy\right\} \qquad (3,30)$$

From the upper equation of (3, 19), $u_n^{(0)}$ is not a function of y, then, Equation (3, 30) leads to

$$\beta^2\left\{\frac{\partial^2\eta_{n-1}^{(1)}}{\partial t^2}-h\frac{\partial^2\eta_{n-1}^{(1)}}{\partial x^2}\right\}=\beta^2\left\{\frac{h^3}{3}\frac{\partial^4\eta_n^{(0)}}{\partial x^4}\right\} \qquad (3,31)$$

where the relation $u_n^{(0)}=\frac{1}{\sqrt{h}}\eta_n^{(0)}$ is used.

If we put

$$\eta-(\kappa')^n\eta_n^{(0)}=\beta(\kappa')^{n-1}\eta_{n-1}^{(1)}$$

, we can obtain with an error of $O(\beta^3)$

$$\beta\left\{\frac{\partial^2\eta}{\partial t^2}-h\frac{\partial^2\eta}{\partial x^2}\right\}=\beta^2\left\{\frac{h^3}{3}\cdot\frac{1}{\kappa'}\frac{\partial^4\eta}{\partial x^4}\right\} \qquad (3,32)$$

Returning to the original variables, (3, 32) is transformed to

$$\frac{\partial^2\eta^*}{\partial t^{*2}}-g^*h^*\frac{\partial^2\eta^*}{\partial x^{*2}}=g^*h^*\frac{\partial^2}{\partial x^{*2}}\left\{\frac{h^{*2}}{3}\frac{\partial^2\eta^*}{\partial x^{*2}}\right\} \qquad (3,33)$$

From (3, 3), (3, 4), we can obtain

$$\frac{\partial^{2n}u_0}{\partial y^{2n}} = (-\beta)^n \frac{\partial^{2n}u_n}{\partial x^{2n}} \tag{3, 34}$$

Since u_0 is not a function of x, (3, 34) satisfies approximately this relation provided a sufficiently large n is taken, for the value of β is smaller than 10^{-2} if $h_0/l_0 < 10^{-1}$, and $(-\beta)^n$ is approaching to zero for a large n. Simultaneously, u_0 is not a function of t, and η_0, u_0 can be considered to be infinitely small disturbances propagating on the surface.

4. Conclusion

As shown in the chapter 2, the fundamental equation to the second order approximation with respect to κ and β, (2, 52), appears in the case $\eta_0{}^* l_0{}^2/h_0{}^3 > 1$. For the case $\eta_0{}^* l_0{}^2/h_0{}^3 < 1$, the second order approximation is the linearized equation as shown by (3, 33). Another complementary equation, the Airy equation, seems to be concerned with the initial values of wave motion.

Taking sinusoidal waves for example, the right-hand side of (2, 33) is considered to be the term amplifying the second higher harmonic component of the original wave, then, this amplifying energy must be supported by any other term which should be located in the right-hand side of (2, 33). Therefore, the Airy equation is not complete in the physical interpretation, and the application of this equation is restricted.

Apart from these statements, if it is assumed the balance between two terms in the right-hand side of (2, 49) are lost, for instance, κ is very small initially, a simple order estimation may be applied. According to the illustrations by Wilson and others, a deformation appears if

$$\frac{\eta_0{}^* l_0{}^2}{h_0{}^3} > 40 \tag{4, 1}$$

This value approximately corresponds to

$$\frac{\dfrac{3}{2}\dfrac{\eta^2}{h}}{\kappa \cdot \dfrac{h^2}{3}\dfrac{\partial^2\eta}{\partial x^2}} > 200 \tag{4, 2}$$

where η, h and $\partial^2\eta/\partial x^2$ are assumed to be of order unity. In the same way, from the third order approximation, (2, 77), if

$$\frac{\dfrac{h}{2}\left(\dfrac{\partial\eta}{\partial x}\right)^2}{\kappa\dfrac{h^4}{30}\dfrac{\partial^4\eta}{\partial x^4}} > 200 \tag{4, 3}$$

that is,

$$\frac{\eta_0{}^* l_0{}^2}{h_0{}^3} > 13, \tag{4, 4}$$

a deformation can appear in the sense of the third order approximation for the wave propagating with the phase velocity nearly equal to $\sqrt{g^*h^*}$, because the equation,

$$\beta\left\{\frac{\partial^2\eta}{\partial t^2}-h\frac{\partial^2\eta}{\partial x^2}\right\}=\beta^2 h\frac{\partial^2}{\partial x^2}\left\{\frac{3}{2}\frac{\eta_s^2}{h}+\kappa\frac{h^2}{3}\frac{\partial^2\eta_s}{\partial x^2}+\beta\kappa\frac{h}{2}\left(\frac{\partial\eta_s}{\partial x}\right)^2\right\} \tag{4, 5}$$

has an approximate solution given by

$$\eta=\eta_s-\frac{h}{2}\beta^2\cdot\kappa\cdot x\cdot\frac{\partial\eta_s}{\partial x}\frac{\partial^2\eta_s}{\partial x^2} \tag{4, 6}$$

where η_s is a solution of (2, 49). Here, note that the deformation is much more slow than that caused by (4, 1) due to β^2 and κ.

Therefore, when $\eta_0^*l_0^2/h_0^3$ is situated in

$$13<\frac{\eta_0^*l_0^2}{h_0^3}<44\,, \tag{4, 7}$$

there is a possibility that the wave form may still satisfy (2, 49), but it may not be strictly a stationary wave.

As mentioned earlier, however, these are concerned with the initial values of wave motion, there still exist difficult and paradoxical problems against the stability of the wave form of long waves.

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