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The length of equivalent fixity of a pile

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1. Introduction

The characteristic number of a pile β is defined as

$$\beta = \sqrt[4]{\frac{k_{CH}D}{4EI}}, \tag{1}$$

where k_{CH} is the modulus of horizontal subgrade reaction, D is the pile diameter and EI is the pile flexural rigidity.

In the design of a marginal wharf, it is sometimes assumed that the pile is fixed at a depth of $1/\beta$ (International Navigation Association, 2001). This means that an actual pile supported by horizontal subgrade reaction is replaced with a hypothetical pile fixed at $1/\beta$. This article describes the basis of this replacement. The central point is that, when the actual pile and the hypothetical pile are subject to the same horizontal load at the top while the deflection angle at the top being kept at zero, they exhibit the same bending moment at the pile top as long as the hypothetical pile is fixed at $1/\beta$. These two piles are equivalent to each other in this sense.

2. Horizontal load – bending moment relation

Figure 1 (left) shows the deflection of an infinite pile supported by horizontal subgrade reaction subject to a horizontal load H at the top. Figure 1 (right) shows the deflection of a pile fixed at $1/\beta$ subject to the same horizontal load H at the top. For the both piles, the deflection angle at the top is kept at zero. When we consider a rigid-frame structure, it could be reasonable to assume that the deflection angle is zero at the top as a first approximation. In Figure 1, the x-axis is pointing downward and $x = 0$ corresponds to the seabed.

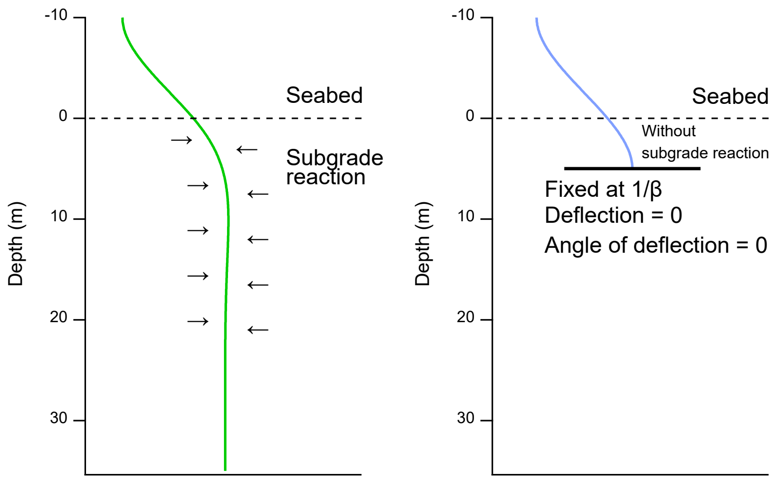

Figure 1 Deflections of an infinite pile supported by horizontal subgrade reaction (left) and a pile fixed at $1/\beta$ (right). In this example, the height above the seabed h=10m and $1/\beta$=5m.

Let us first consider the governing equation for the deflection w of the embedded portion ($x > 0$) of the pile supported by horizontal subgrade reaction (Figure 1 left). The external force per unit length q can be written as

$$q = -k_{CH} D w \tag{2}$$

and the governing equation is

$$EI \frac{\partial^4 w}{\partial x^4} + k_{CH} D w = 0. \tag{3}$$

By using the definition of β in Equation (1), the governing equation becomes

$$\frac{\partial^4 w}{\partial x^4} + 4\beta^4 w = 0 \tag{4}$$

and the general solution is given by

$$w = C_1 e^{(1+i)\beta x} + C_2 e^{(1-i)\beta x} + C_3 e^{(-1+i)\beta x} + C_4 e^{(-1-i)\beta x}. \tag{5}$$

Because the deflection should not tend to infinity as $x \to \infty$, $C_1 = C_2 = 0$. Therefore, the general solution is given by

$$w = C_3 e^{(-1+i)\beta x} + C_4 e^{(-1-i)\beta x} \tag{6}$$

$$w' = \beta \left((-1+i) C_3 e^{(-1+i)\beta x} + (-1-i) C_4 e^{(-1-i)\beta x} \right) \tag{7}$$

$$w'' = \beta^2 \left((-1+i)^2 C_3 e^{(-1+i)\beta x} + (-1-i)^2 C_4 e^{(-1-i)\beta x} \right) \tag{8}$$

$$w''' = \beta^3 \left((-1+i)^3 C_3 e^{(-1+i)\beta x} + (-1-i)^3 C_4 e^{(-1-i)\beta x} \right). \tag{9}$$

On the other hand, the governing equation for the deflection w of the pile above the seabed is

$$\frac{\partial^4 w}{\partial x^4} = 0 \tag{10}$$

and the general solution is given by

$$w = A_1 x^3 + A_2 x^2 + A_3 x + A_4 \tag{11}$$

$$w' = 3A_1 x^2 + 2A_2 x + A_3 \tag{12}$$

$$w'' = 6A_1 x + 2A_2 \tag{13}$$

$$w''' = 6A_1. \tag{14}$$

Equations (6) – (9) and (11) – (14) involve 6 undetermined constants and they are determined from the continuity conditions at the seabed ($x = 0$) and the boundary conditions at the pile top ($x = -h$). From the continuity conditions at the seabed ($x = 0$) we have

$$C_3 + C_4 = A_4 \tag{15}$$

$$\beta \left((-1+i) C_3 + (-1-i) C_4 \right) = A_3 \tag{16}$$

$$\beta^2((-1+i)^2C_3+(-1-i)^2C_4)=2A_2 \tag{17}$$
$$\beta^3((-1+i)^3C_3+(-1-i)^3C_4)=6A_1. \tag{18}$$

From the boundary conditions at the pile top ($x=-h$) we have

$$3A_1h^2-2A_2h+A_3=0 \tag{19}$$
$$-EI\cdot 6A_1=-H. \tag{20}$$

From Equations (15) – (20) we obtain

$$A_1=\frac{H}{6EI} \tag{21}$$

$$A_2=\frac{(h\beta-1)H}{4EI\beta} \tag{22}$$

$$A_3=-\frac{hH}{2EI\beta} \tag{23}$$

$$A_4=\frac{(h\beta+1)H}{4EI\beta^3} \tag{24}$$

$$C_3=\frac{(h\beta+1)H}{8EI\beta^3}+i\frac{(h\beta-1)H}{8EI\beta^3} \tag{25}$$

$$C_4=\frac{(h\beta+1)H}{8EI\beta^3}-i\frac{(h\beta-1)H}{8EI\beta^3}. \tag{26}$$

From Equation (13), the bending moment at the pile top is

$$M=\frac{1}{2}\left(h+\frac{1}{\beta}\right)H. \tag{27}$$

It should be noted that, because C_3 and C_4 are conjugate of each other, the righthand side of Equation (6) is a sum of two complex numbers that are conjugate of each other. Therefore, the deflection is real.

Next, for the pile fixed at $1/\beta$ (Figure 1 right), the moment at the pile top is equal to the moment at the fixed point due to symmetry. Therefore, from the equilibrium of moment, we have

$$M=-M+\left(h+\frac{1}{\beta}\right)H \tag{28}$$

and therefore

$$M=\frac{1}{2}\left(h+\frac{1}{\beta}\right)H. \tag{29}$$

Equations (27) and (29) are identical. It means that, when the pile supported by horizontal subgrade reaction (Figure 1 left) and the pile fixed at $1/\beta$ (Figure 1 right) are subject to the same horizontal load at the top while the deflection angle at the top being kept at zero, the same bending moment is induced at the pile top.

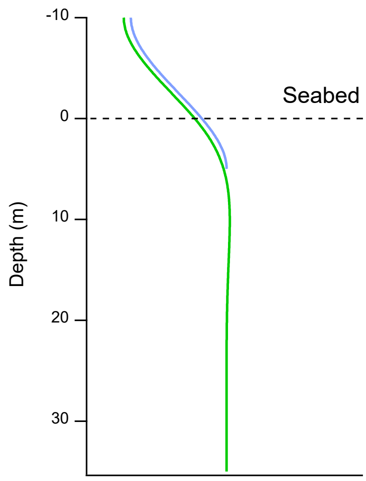

Figure 2 Comparison of the deflection curves of the infinite pile supported by horizontal subgrade reaction (green) and the pile fixed at $1/\beta$ (blue). In this example, the height above the seabed h=10m and $1/\beta$=5m.

Figure 2 shows a comparison of the deflection curves of the infinite pile supported by horizontal subgrade reaction and the pile fixed at $1/\beta$ subject to the same horizontal load. The latter curve is a good approximation for the former curve. The displacement is slightly smaller for the latter curve.

3. Comparison of natural frequency

So far, it was found that the infinite pile supported by horizontal subgrade reaction and the pile fixed at $1/\beta$ are equivalent to each other in a sense that, when they are subject to the same horizontal load at the top while the deflection angle at the top being kept at zero, they exhibit the same bending moment at the pile top. These two piles are equivalent to each other in this sense. Next, we will investigate how close the natural frequencies are for the structures supported by these two piles.

We assume a concentrated mass M at the pile top and neglect distributed masses. Then the governing equations are the same as those for the static cases.

The governing equation for the deflection w of the embedded portion ($x > 0$) of the pile supported by horizontal subgrade reaction (Figure 1 left) is the same as Equation (4) and it is

$$\frac{\partial^4 w}{\partial x^4} + 4\beta^4 w = 0. \tag{30}$$

Fourier transform of this equation yields

$$\frac{\partial^4 \hat{w}}{\partial x^4} + 4\beta^4 \hat{w} = 0 \tag{31}$$

and the general solution is given by

$$\hat{w} = C_1 e^{(1+i)\beta x} + C_2 e^{(1-i)\beta x} + C_3 e^{(-1+i)\beta x} + C_4 e^{(-1-i)\beta x} \tag{32}$$

Because the deflection should not tend to infinity as $x \to \infty$, $C_1 = C_2 = 0$. Therefore, the general solution is given by

$$\hat{w} = C_3 e^{(-1+i)\beta x} + C_4 e^{(-1-i)\beta x} \tag{33}$$

$$\hat{w}' = \beta\left((-1+i)C_3 e^{(-1+i)\beta x} + (-1-i)C_4 e^{(-1-i)\beta x}\right) \tag{34}$$

$$\hat{w}'' = \beta^2\left((-1+i)^2 C_3 e^{(-1+i)\beta x} + (-1-i)^2 C_4 e^{(-1-i)\beta x}\right) \tag{35}$$

$$\hat{w}''' = \beta^3\left((-1+i)^3 C_3 e^{(-1+i)\beta x} + (-1-i)^3 C_4 e^{(-1-i)\beta x}\right). \tag{36}$$

On the other hand, the governing equation for the deflection w of the pile above the seabed is the same as Equation (10) and it is

$$\frac{\partial^4 w}{\partial x^4} = 0. \tag{37}$$

Fourier transform of this equation yields

$$\frac{\partial^4 \hat{w}}{\partial x^4} = 0 \tag{38}$$

and the general solution is given by

$$\hat{w} = A_1 x^3 + A_2 x^2 + A_3 x + A_4 \tag{39}$$

$$\hat{w}' = 3A_1 x^2 + 2A_2 x + A_3 \tag{40}$$

$$\hat{w}'' = 6A_1 x + 2A_2 \tag{41}$$

$$\hat{w}''' = 6A_1. \tag{42}$$

The continuity conditions at the seabed ($x = 0$) and the boundary conditions at the pile top ($x = -h$) are exactly the same as Equations (15) – (20) except that H should be replaced by its Fourier transform $\hat{H}$. Therefore, A_1 – A_4 is given by Equations (21) – (24) except that H should be replaced by its Fourier transform $\hat{H}$. The horizontal load in this case is due to the inertia force and it is given by

$$H = -M\ddot{w}|_{x=-h}. \tag{43}$$

Fourier transform of this equation yields

$$\hat{H} = \omega^2 M \hat{w}|_{x=-h}. \tag{44}$$

Substituting Equations (39)(21)(22)(23)(24) into Equation (44) and replacing H with $\hat{H}$ yields

$$\hat{H} = \omega^2 \frac{M}{12EI} \frac{h^3\beta^3 + 3h^2\beta^2 + 3h\beta + 3}{\beta^3} \hat{H}. \tag{45}$$

By requiring that this equation has a nontrivial solution, the natural frequency can be determined as

$$\omega = \sqrt{\frac{12EI}{\left[\left(h+\frac{1}{\beta}\right)^3+\frac{2}{\beta^3}\right]M}}. \tag{46}$$

Next, we will consider the pile fixed at $1/\beta$ (Figure 1 right). For simplicity, let $x = 0$ at the pile top. By assuming that $w' = 0$ both at the pile top and the fixed point and $w = 0$ at the fixed point, the deflection and its derivatives can be given by

$$\widehat{w}''' = \frac{\widehat{H}}{EI} \tag{47}$$

$$\widehat{w}'' = \frac{\widehat{H}}{EI}\left(x - \frac{1}{2}\left(h + \frac{1}{\beta}\right)\right) \tag{48}$$

$$\widehat{w}' = \frac{\widehat{H}}{EI}\left(\frac{x^2}{2} - \frac{1}{2}\left(h + \frac{1}{\beta}\right)x\right) \tag{49}$$

$$\widehat{w} = \frac{\widehat{H}}{EI}\left(\frac{x^3}{6} - \frac{1}{4}\left(h + \frac{1}{\beta}\right)x^2 + \frac{1}{12}\left(h + \frac{1}{\beta}\right)^3\right). \tag{50}$$

The horizontal load in this case is due to the inertia force and it is given by

$$H = -M\ddot{w}|_{x=0}. \tag{51}$$

Fourier transform of this equation yields

$$\widehat{H} = \omega^2 M\widehat{w}|_{x=0}. \tag{52}$$

Substituting Equation (50) into Equation (52) yields

$$\widehat{H} = \omega^2 \frac{M}{12EI}\left(h + \frac{1}{\beta}\right)^3 \widehat{H}. \tag{53}$$

By requiring that this equation has a nontrivial solution, the natural frequency can be determined as

$$\omega = \sqrt{\frac{12EI}{\left(h+\frac{1}{\beta}\right)^3 M}}. \tag{54}$$

By comparing Equations (46) and (54), it can be found that the natural frequencies are close to each other. However, the infinite pile supported by horizontal subgrade reaction has a slightly lower natural frequency than the pile fixed at $1/\beta$. This corresponds to the slight difference of the deflection curves shown in Figure 2.

References

International Navigation Association (2001): Seismic Design Guidelines for Port Structures, p.328.