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REPORT OF THE PORT AND
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1. The Problems of Density Current Part II

Tokuichi HAMADA

Synopsis

This paper follows on "The problems of density current. Part I. (1967)", and consists of two chapters.

In chapter 1, the stability problems of interfacial and internal waves are discussed. At first the property of the wave-induced Reynolds stress at the Kelvin-Helmholtz instability is examined. The present method is more general than in Part I. The result shows that the Reynolds stress in this case can be treated as a particular one of ordinary treatment of instability regulated by the second derivative of velocity profile of the general flow.

Secondly the instability of interfacial wave at a two-layer flow is treated. The velocity profile of the upper layer is assumed parabolic. To solve the eigen value equation easily, a simplified method is used. Two numerical examples are recorded, and the results are agreeable. In the appendix the change of wave celerity of internal wave in a case, in which a mixed layer between two homogeneous layers exists, is examined.

Thirdly the instability of internal wave with a shear flow is studied. The variation of fluid density is same as in the above-mentioned appendix. The result shows the noticeable effect of the velocity profile of shear flow in the upper homogeneous layer against the stability.

In chapter 2, the problem of control section of two-layer flows is discussed. Firstly a case of the linear long wave is treated, and secondly the problem of interfacial hydraulic jump is examined. Thirdly the interfacial resistance and the variation of width of flow are taken into account. The sesult shows a new aspect of this problem, and it may be applicable to some actual events.

港湾技術研究所報告
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1. 密度流の問題 II

浜　田　徳　一

要　旨

この研究は'密度流の問題，そのI (1967)'につづくものであり，2章からなりたっている。

第1章においては，2層流体の界面内波および成層流体の内波の問題がとり扱われる。界面内波における不安定 Reynolds 応力の性質，不安定の形成，混合層の存する場合の内波の波速の変化，および成層流体の内波の安定が，これにかかる剪断流の流速分布により支配せられることなどが示されている。

第2章においては，2層流の制御断面の性質が検討せられる。微小振幅の場合，ジャンプの場合および界面抵抗，幅の変化を考慮した場合と漸次複雑なものえと進んでいる。界面抵抗，水流幅の変化を考慮した場合には新しい視野が現われ，応用面においても有効であろうと考えられる。

CONTENTS

1. On the Stability Problems of Interfacial and Internal Waves

1-1. Wave-induced Reynolds stress of the Kelvin-Helmholtz instability

In the Kelvin-Helmholtz instability of two layer fluids with uniform flows discontinuous at the interface, the Reynolds stress acts on both the surface and the back of the interfacial boundary, and the stress on the back is equal and opposite to the stress on the surface. The increase of momentum and mechanical energy of instability wave is controlled reasonably by this Reynolds stress. This is already stated in "The problems of density current, Part I" (T. Hamada (1967)). Here another aspect of this Reynolds stress is explained to make clear the property in the relation with the more general view of the wave-induced Reynolds stress.

x-axis (abscissa) is taken horizontally along the interfacial boundary in still condition, and y-axis (ordinate) is taken vertically upwards positive. In the reference of section 3.1 and 3.2 of Part I, we put the vertical gradient of density and the no-perturbed flow as

$$\bar{\rho}_y=(\bar{\rho}^{(1)}-\bar{\rho}^{(2)})\delta(y), \quad U_y=(U^{(1)}-U^{(2)})\delta(y) \tag{1-1}$$

At the same time, from the relation (3-15) of Part I, the real part of the complex phase celerity $c=c_r+ic_i$ can be written in its first order approximation

$$c_r=\frac{\bar{\rho}^{(1)}U^{(1)}+\bar{\rho}^{(2)}U^{(2)}}{\bar{\rho}^{(1)}+\bar{\rho}^{(2)}} \tag{1-2}$$

and c_r of this expression is also considered to coincide with the velocity of no-perturbed flow just at the interface. Here $\bar{\rho}^{(1)}$, $U^{(1)}$... are concerned to the upper fluid, and $\bar{\rho}^{(2)}$, $U^{(2)}$, ... are related to the lower fluid, and the suffix y in (1-1) means the differentiation with respect to y.

The perturbed motion is assumed two-dimensional. The stream function of the perturbed wave motion may be expressed by

$$\psi=\varphi(y)e^{ik(x-ct)}, \quad u=-\psi_y \quad \text{and} \quad v=\psi_x \tag{1-3}$$

u and v means the horizontal and the vertical velocity respectively, and here $\varphi(y)$ and c are complex. We consider the perturbed motion of vertically stratified fluid with inviscid property (or more practically the Reynolds number of the fluid motion is very large), and so the motion of the first order is expressed by

$$\varphi_{yy}+\frac{\bar{\rho}_y}{\bar{\rho}}\varphi_y-\left\{k^2-\frac{U_{yy}}{c_r-U+ic_i}+\frac{g}{\bar{\rho}}\frac{\bar{\rho}_y}{(c_r-U+ic_i)^2}-\frac{\bar{\rho}_y}{\bar{\rho}}\frac{U_y}{c_r-U+ic_i}\right\}\varphi=0 \tag{1-4}$$

J.W. Miles (1961) may be referred to introduce this relation. In (1-4) U is horizontal general flow with vertical shear, and if $\bar{\rho}_y=0$, this relation comes back (4.3.1) of C.C. Lin (1955). In the present problem of instability, the characteristic equation was solved by another way. c_r was already given by (1-2), and $c_i>0$ was also established.

Then we consider the wave-induced Reynolds stress, which is kept by the product of horizontal and vertical velocities of the perturbed wave motion at a fixed point. In general

$$\tau = -\frac{1}{2}\boldsymbol{R}(\bar{\rho}\,\overline{uv^*}) \tag{1-5}$$

Here asterisk means the complex conjugate, and $\boldsymbol{R}(\quad)$ means the real part of $(\quad)$. (1-5) is transformed by making use of relations of (1-3).

$$\tau = \frac{\bar{\rho}k}{4i}(\varphi_y\varphi^* - \varphi_y^*\varphi)e^{2kc_i t} \tag{1-6}$$

On the other hand, from (1-4)

$$\varphi_{yy} + \frac{\bar{\rho}_y}{\bar{\rho}}\varphi_y - \left\{k^2 - \frac{(\bar{\rho}U_y)_y}{\bar{\rho}}\frac{c_r - U}{(c_r - U)^2 + c_i^2} + \frac{g}{\bar{\rho}}\frac{\bar{\rho}_y\{(c_r - U)^2 - c_i^2\}}{\{(c_r - U)^2 - c_i^2\}^2 + 4(c_r - U)^2 c_i^2}\right\}\varphi$$
$$-i\left\{\frac{(\bar{\rho}U_y)_y c_i}{\bar{\rho}(c_r - U)^2 + c_i^2} - \frac{g}{\bar{\rho}}\frac{2(c_r - U)c_i}{\{(c_r - U)^2 - c_i^2\}^2 + 4(c_r - U)^2 c_i^2}\right\}\varphi = 0 \tag{1-7}$$

If we multiply (1-7) by φ^*, the complex conjugate of φ, and subtract its complex conjugate from the resulted relation, we have

$$\varphi_{yy}\varphi^* - \varphi\varphi_{yy}^* + \frac{\bar{\rho}_y}{\bar{\rho}}(\varphi_y\varphi^* - \varphi\varphi_y^*) - 2i\left\{\frac{(\bar{\rho}U_y)_y c_i}{\bar{\rho}\{(c_r - U)^2 + c_i^2\}}\right.$$
$$\left. - \frac{g}{\bar{\rho}}\frac{2(c_r - U)c_i}{\{(c_r - U)^2 - c_i^2\}^2 + 4(c_r - U)^2 c_i^2}\right\}\varphi\varphi^* = 0 \tag{1-8}$$

Properties of wave motion of the present instability are used in (1-8). As the motion is irrotational in both fluids, the Reynolds stress does not appear apart from the interface $y=0$. So the treatment is limited to the vicinity of the interface. We integrate (1-8) from $y=-\Delta$ to $y=+\Delta$, (Δ is a positive small quantity), and so the path of integral passes the interface.

$$\int_{-\Delta}^{\Delta}(\varphi_{yy}\varphi^* - \varphi\varphi_{yy}^*)dy = \left|\varphi_y\varphi^* - \varphi\varphi_y^*\right|_{-\Delta}^{\Delta} \tag{1-9}$$

(1-9) is a result of partial integral. In comparison with (1-6), this term is not active because $\tau=0$ at $y=\pm\Delta$. Then

$$\int_{-\Delta}^{\Delta}\frac{\bar{\rho}_y}{\bar{\rho}}(\varphi_y\varphi^* - \varphi\varphi_y^*)dy = \frac{\bar{\rho}^{(1)} - \bar{\rho}^{(2)}}{\bar{\rho}^{(0-1)}}(\varphi_y\varphi^* - \varphi\varphi_y^*)_0 \tag{1-10}$$

In this modification the expression of $\bar{\rho}_y$ of (1-1) is used. $\bar{\rho}^{(0-1)}$ has an intermediate value of $\bar{\rho}^{(2)}$ and $\bar{\rho}^{(1)}$, but is not determined here. In concern with the term of $-2i\{\cdots\}\varphi\varphi^*$ of (1-8), the integral of the part concerned to the second term in the brackets may be neglected, if Δ is sufficiently small. We use the following approximation for the integral concerned to the first term of the brackets.

$$\bar{\rho}U_y = \bar{\rho}^{(0)}(U^{(1)} - U^{(2)})\delta(y) \tag{1-11}$$

and then

$$\int_{-\Delta}^{\Delta}\frac{(\bar{\rho}U_y)_y c_i}{\bar{\rho}\{(c_r - U)^2 + c_i^2\}}\varphi\varphi^* dy = \int_{-\Delta}^{\Delta}(U^{(1)} - U^{(2)})\frac{d\delta(y)}{dy}\frac{c_i}{\{(c_r - U)^2 + c_i^2\}}|\varphi|^2 dy$$
$$= \alpha k(U^{(1)} - U^{(2)})\frac{1}{c_i}\left|\varphi\right|_{-0}^{2} - \alpha k(U^{(1)} - U^{(2)})\frac{1}{c_i}\left|\varphi\right|_{+0}^{2} \tag{1-12}$$

Here α is a positive constant usually of 10^0 order. The solution of the first order approximation of the present problem of instability is shown in section 3.1 and 3.2 of Part I. According to this solution, $|\varphi|^2_{-0}=|\varphi|^2_{+0}=|\varphi|^2_0$ is consistent, and by making use of the wave amplitude A_0 at the interface

$$|\varphi|^2_0=A_0^2c_i^2 \tag{1-13}$$

In consideration of the property of $d\delta(y)/dy$, we can see ahat the first term of (1-12) is active at the lower surface of the interfacial plane, and that the second term is for its upper surface.

From (1-10), (1-12) and (1-13),

$$\begin{aligned}\frac{\bar{\rho}^{(1)}-\bar{\rho}^{(2)}}{\bar{\rho}^{(0-1)}}(\varphi_y\varphi^*-\varphi\varphi_y^*)_0&=2i\alpha k(U^{(1)}-U^{(2)})c_iA_0^2 \quad \text{at} \quad y=-0\\ &=-2i\alpha k(U^{(1)}-U^{(2)})c_iA_0^2 \quad \text{at} \quad y=+0\end{aligned} \tag{1-14}$$

In the reference of (1-6), the Reynolds stress at the interface is

$$\left.\begin{aligned}\tau_{(-0)}&=\frac{\alpha}{2}\frac{\bar{\rho}^{(0-1)}\bar{\rho}^{(0-2)}}{\bar{\rho}^{(1)}-\bar{\rho}^{(2)}}k^2(U^{(1)}-U^{(2)})c_iA_0^2e^{2kc_it}\\ \tau_{(+0)}&=-\frac{\alpha}{2}\frac{\bar{\rho}^{(0-1)}\bar{\rho}^{(0-2)}}{\bar{\rho}^{(1)}-\bar{\rho}^{(2)}}k^2(U^{(1)}-U^{(2)})c_iA_0^2e^{2kc_it}\end{aligned}\right\} \tag{1-15}$$

As $\bar{\rho}^{(2)}>\bar{\rho}^{(1)}$, if $U^{(1)}>U^{(2)}$, $\tau_{(-0)}$ acts to the negative direction, and $\tau_{(+0)}$ is for the positive direction.

In (1-15) a coefficient $\alpha\bar{\rho}^{(0-1)}\bar{\rho}^{(0-2)}$ is not determined by the present procedures, and we go back to the treatment of Part I. The increase rate of the momentum transport corresponding to the Reynolds stress of (1-15) was shown by relation (3-20) and (3-21) of Part I.

$$\left.\begin{aligned}\frac{dM^{(1)}}{dt}&=-A_0^2k^2c_i\bar{\rho}^{(1)}\bar{\rho}^{(2)}\frac{U^{(1)}-U^{(2)}}{\bar{\rho}^{(1)}+\bar{\rho}^{(2)}}e^{2kc_it}\\ \frac{dM^{(2)}}{dt}&=A_0^2k^2c_i\bar{\rho}^{(1)}\bar{\rho}^{(2)}\frac{U^{(1)}-U^{(2)}}{\bar{\rho}^{(1)}+\bar{\rho}^{(2)}}e^{2kc_it}\end{aligned}\right\} \tag{1-16}$$

From this,

$$\frac{\alpha}{2}\frac{\bar{\rho}^{(0-1)}\bar{\rho}^{(0-2)}}{\bar{\rho}^{(1)}-\bar{\rho}^{(2)}}=-\frac{\bar{\rho}^{(1)}\bar{\rho}^{(2)}}{\bar{\rho}^{(1)}+\bar{\rho}^{(2)}} \tag{1-17}$$

By this way the Reynolds stress is determined, and the expression is mainly regulated by $U_{yy}=(U^{(1)}-U^{(2)})\dfrac{d\delta(y)}{dy}$ of (1-12). Stillmore $c_r=U^{(i)}$ ((i) means the interface) is acceptable at $y=0$ in the present case. From these properties we can understand the Reynolds stress in the present case as one of special cases of the expression of wave-induced Reynolds stress shown by the relation (4.3.6.) of C.C. Lin (1955, p. 54). The form (3-24) of Part I was an intuitive introduction of thisstress, and the present form of the introduction may clarify its relation to the more general case.

1-2. An instability analysis of interfacial wave caused by a shear flow

Here we investigate an instability of interfacial waves in stable two-layer fluids caused by shear flow with the velocity profile of moderate curvature. In general the viscous term of the perturbed equation is not negligible in these stability problems. But according to the analysis of T.B. Benjamin (1959), if the height of the critical layer from the wavy boundary is far greater than the thickness of so called "wall friction layer", the inviscid solution of interfacial wave offers a good approximation. Using this idea clarified, the present treatment is assumed inviscid. We use the co-ordinate system of 1-1. $y=0$ is taken at the interface in still condition. $U^{(1)}$, $u^{(1)}$, $v^{(1)}$, $\rho^{(1)}$.... are concerned to the upper fluid, and $U^{(2)}$, $u^{(2)}$, $v^{(2)}$, $\rho^{(2)}$.... are for the lower fluid. The upper boundary of the upper fluid is taken horizontally at $y=h_1$, and the horizontal lower boundary

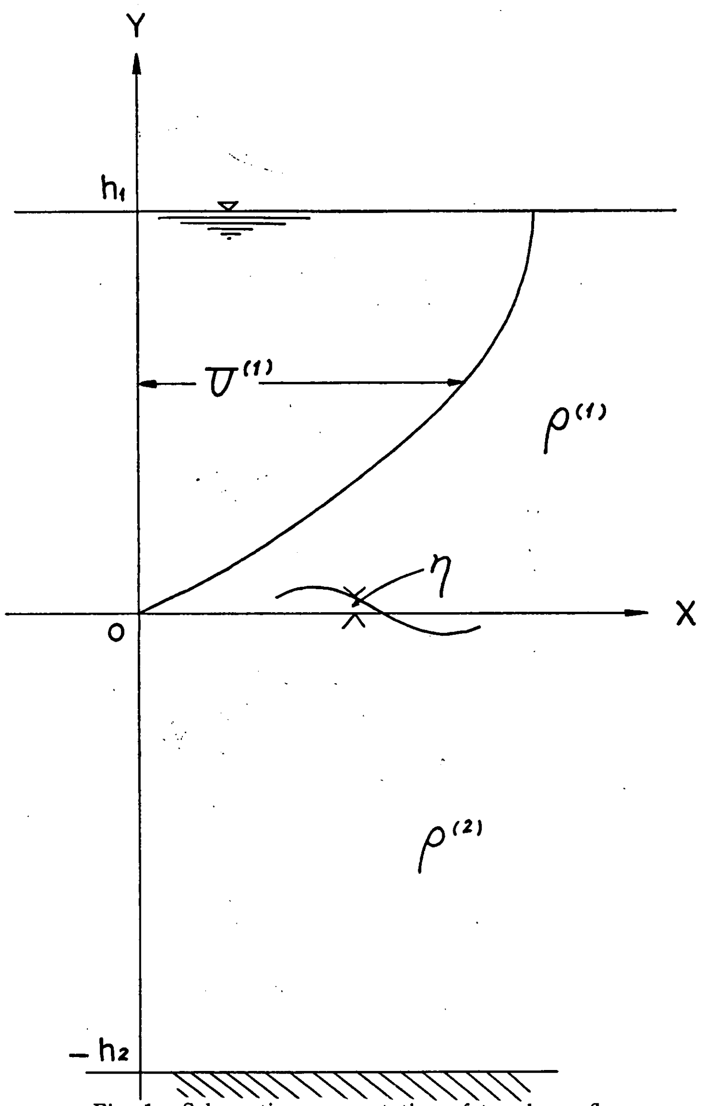

Fig. 1. Schematic representation of two layer flow

of the lower fluid is at $y=-h_2$. To simplify the problem, we assume $U^{(1)}$ has a profile of steady parabolic type, and $U^{(2)}=0$. (Fig. 1)

$$U^{(1)}=\frac{U^{(1-s)}}{h_1^2}y(2h_1-y), \quad U^{(2)}=0 \tag{1-18}$$

$y=h_1$ means a free surface, and $U^{(1-s)}$ is a velocity of the general flow at the free surface.

In the same way of (1-3) and (1-4), for the upper fluid

$$(U^{(1)}-c)(\varphi_{yy}^{(1)}-k^2\varphi^{(1)})-U_{yy}^{(1)}\varphi^{(1)}=0 \tag{1-19}$$

for the lower fluid

$$\varphi_{yy}^{(2)}-k^2\varphi^{(2)}=0 \tag{1-20}$$

Interfacial boundary conditions are

$$P^{(1)}+p^{(1)}=P^{(2)}+p^{(2)} \qquad \text{at} \quad y=\eta \tag{1-21}$$

$$\left.\begin{aligned} &\frac{\partial\eta}{\partial t}+U^{(1)}\frac{\partial\eta}{\partial x}=v^{(1)} \qquad \text{at} \quad y=\eta \\ &\frac{\partial\eta}{\partial t}=v^{(2)} \end{aligned}\right\} \tag{1-22}$$

$P^{(1)}$ and $P^{(2)}$ are from no-perturbed condition, and by making use of

$$U^{(1)}(\eta)=U^{(1)}(0)+U_y^{(1)}(0)\eta=U_y^{(1)}(0)\eta$$

the simplified linear relation from (1-22) is

$$\frac{\partial\eta}{\partial t}=v^{(1)}, \quad \frac{\partial\eta}{\partial t}=v^{(2)} \qquad \text{at} \quad y=0 \tag{1-23}$$

Boundary conditions at $y=h_1$ and $-h_2$ are

$$\left.\begin{aligned} v^{(1)}=0 \quad \text{at} \quad y=h_1 \\ v^{(2)}=0 \quad \text{at} \quad y=-h_2 \end{aligned}\right\} \tag{1-24}$$

The boundary condition at $y=-h_2$ is correct at a rigid inviscid boundary, but the condition at $y=h_1$ is an approximation with respect to the upper free surface of interfacial wave.

We select the method of W. Heisenberg (C.C. Lin (1945, 1955)) for an approximate solution of (1-19). This method is usually used in a case in which h_1 is finite, and stillmore, as will be shown later, it is conveniently extended to the approximate estimation, when h_1 is large. By this method, using the boundary condition at $y=h_1$,

$$\begin{aligned} \varphi^{(1)}(y)=&-(U^{(1)}-c)B^{(1)}F_2\left\{1+k^2\int_0^y(U^{(1)}-c)^{-2}\int_0^y(U^{(1)}-c)^2dydy+\cdots\right\} \\ &+(U^{(1)}-c)B^{(1)}\left\{\int_0^y(U^{(1)}-c)^{-2}dy+k^2\int_0^y(U^{(1)}-c)^{-2}\right. \\ &\left.\times\int_0^y(U^{(1)}-c)^2\int_0^y(U^{(1)}-c)^{-2}dydydy+\cdots\right\} \end{aligned} \tag{1-25}$$

$$F_2=\frac{\int_0^{h_1}(U^{(1)}-c)^{-2}dy+k^2\int_0^{h_1}(U^{(1)}-c)^{-2}\int_0^y(U^{(1)}-c)^2\int_0^y(U^{(1)}-c)^{-2}dydydy+\cdots\cdot}{1+k^2\int_0^{h_1}(U^{(1)}-c)^{-2}\int_0^y(U^{(1)}-c)^2dydy+\cdots\cdot} \tag{1-26}$$

Here $B^{(1)}$ is an arbitrary constant. Using the amplitude $A^{(0)}$ of interfacial wave at the interface,

$$\eta=A^{(0)}e^{ik(x-ct)} \tag{1-27}$$

$$\varphi^{(1)}(y)=(U^{(1)}-c)A^{(0)}\left\{1+k^2\int_0^y(U^{(1)}-c)^{-2}\int_0^y(U^{(1)}-c)^2\,dydy+\cdots\cdot\right\}$$
$$-(U^{(1)}-c)\frac{A^{(0)}}{F_2}\left\{\int_0^y(U^{(1)}-c)^{-2}dy+k^2\int_0^y(U^{(1)}-c)^{-2}\int_0^y(U^{(1)}-c)^2\right.$$
$$\left.\times\int_0^y(U^{(1)}-c)^{-2}dydydy+\cdots\cdot\right\} \tag{1-28}$$

Using this expression, $P^{(1)}+p^{(1)}$ at the interface $y=\eta$ is given by

$$P^{(1)}+p^{(1)}=-\rho^{(1)}\frac{A^{(0)}}{F_2}e^{ik(x-ct)}-\rho^{(1)}gA^{(0)}e^{ik(x-ct)} \tag{1-29}$$

On the other hand the solution of (1–20), which satisfies the boundary condition at $y=-h_2$, is given by

$$\varphi^{(2)}=D^{(2)}\sinh k(y+h_2)$$

In consideration of interfacial displacement of (1–27),

$$\varphi^{(2)}=\frac{-cA^{(0)}}{\sinh kh_2}\sinh k(y+h_2) \tag{1-30}$$

From this, $P^{(2)}+p^{(2)}$ at the interface $y=\eta$ is

$$P^{(2)}+p^{(2)}=\rho^{(2)}c^2A^{(0)}k\coth kh_2e^{ik(x-ct)}-\rho^{(2)}gA^{(0)}e^{ik(x-ct)} \tag{1-31}$$

A characteristic equation is obtained from (1–29) and (1–31).

$$(\rho^{(2)}-\rho^{(1)})g=\rho^{(2)}c^2k\coth kh_2+\rho^{(1)}\frac{1}{F_2} \tag{1-32}$$

c in (1–32) is usually complex, but, if $U^{(1)}=0$, (1–26) is simplified to

$$\frac{1}{F_2}=c^2k\coth kh_1$$

In this case (1–32) is

$$(\rho^{(2)}-\rho^{(1)})g=c^2k(\rho^{(2)}\coth kh_2+\rho^{(1)}\coth kh_1) \tag{1-33}$$

This is a well-known characteristic equation of interfacial wave without current.

We consider the simplified approximate method to solve (1–32). In general the first term of F_2 in (1–26) can be expressed by $\int_{y_1}^{y_2}(U^{(1)}-c)^{-2}dy$. Here y_1 and y_2 are real, and the path of the integral proceeds on the real aixs. We take the limit of $c_i\to+0$ at $c=c_r+ic_i$. $U_y^{(1)}>0$ is clear, and taking the case of $y_2>y_{c_r}>y_1$ at $U^{(1)}(y_{c_r})=c_r$, we have approximately.

$$\int_{y_1}^{y_2}\frac{dy}{(U^{(1)}-c)^2}=\frac{1}{U_c^{(1)\prime 2}}\left(-\frac{1}{y_2-y_{c_r}}+\frac{1}{y_1-y_{c_r}}\right)$$

$$-\frac{U_c^{(1)\prime\prime}}{U_c^{(1)\prime 3}}\log_e|y_2-y_{c_r}|+\frac{U_c^{(1)\prime\prime}}{U_c^{(1)\prime 3}}\log_e|y_1-y_{c_r}|-\frac{U_c^{(1)\prime\prime}}{U_c^{(1)\prime 3}}i\pi \qquad (1\text{–}34)$$

Here $U_c^{(1)\prime}$ and $U_c^{(1)\prime\prime}$ are the first and second derivative of $U^{(1)}$ with respect to y at $y=y_{c_r}$ respectively. When y_{c_r} approaches y_1 in (1–34), the real part is approximately expressed by $\frac{1}{U_c^{(1)\prime 2}}\left(-\frac{1}{y_2-y_{c_r}}+\frac{1}{y_1-y_{c_r}}\right)$. This is negative, and is controlled by the first derivative of $U^{(1)}$ at y_{c_r}. On the other hand the imaginary part of (1–34) is controlled by the second derivative of $U^{(1)}$ at y_{c_r}. By this way at the case of $k\to0$ the real part of F_2 and also the real part of (1–32) can be treated using the first derivative of $U^{(1)}$ only. We enlarge this method to the general case of (1–32). It means that, at the complex solution of $c=c_r+ic_i$ of (1–32), c_i is controlled by the second derivative of $U^{(1)}$ at y_{c_r}, and $U_c^{(1)\prime\prime}$ should be in consideration, but that the real part c_r is approximated by the solution which is obtained by the case of linear profile of $U^{(1)}$. Then we consider the flow of the triangular velocity profile, which has the same total discharge with the flow of upper layer of (1–18) (at $y=h_1$, $U^{(1)}=\frac{4}{3}U^{(1-s)}$; at $y=0$, $U^{(1)}=0$ in this triangular velocity profile). Its vertical gradient of velocity equals the first derivative of $U^{(1)}$ of (1–18) at $y=\frac{1}{3}h_1$. In the present case usual value of y_{c_r} is smaller than $\frac{1}{3}h_1$, and we use the flow of triangular velocity profile to find c_r approximately.

Putting as follows in (1–32),

$$\frac{1}{F_2}=Ac^2+Bc+D \qquad (1\text{–}35)$$

for the above-mentioned triangular velocity profile of $U^{(1)}$, we obtain

$$A=k\coth kh_1,\quad B=-\alpha,\quad D=0 \qquad (1\text{–}36)$$

Here $U^{(1)}=\alpha y$ is assumed. By this way

$$\frac{1}{F_2}=c^2k\coth kh_1-\alpha c \qquad (1\text{–}37)$$

Hereupon we take F_2 and c as complex variables. Putting $F_2\to F_{2r}+iF_{2i}$, $c\to c_r+ic_i$ in (1–32), we have

$$(\rho^{(2)}-\rho^{(1)})g=\rho^{(2)}(c_r+ic_i)^2k\coth kh_2+\rho^{(1)}(c_r+ic_i)^2k\coth kh_1$$

$$-\rho^{(1)}\alpha(c_r+ic_i)-\rho^{(1)}i\frac{F_{2i}}{F_{2r}^2} \tag{1-38}$$

In (1-38) $|F_{2r}|\gg|F_{2i}|$ is assumed. The imaginary part of (1-38) is

$$\begin{aligned} c_i&=\rho^{(1)}\frac{F_{2i}}{F_{2r}^2}\frac{1}{2\rho^{(2)}c_rk\coth kh_2+2\rho^{(1)}c_rk\coth kh_1-\rho^{(1)}\alpha} \\ &=\rho^{(1)}F_{2i}\frac{c_r^2(c_rk\coth kh_1-\alpha)^2}{2\rho^{(2)}c_rk\coth kh_2+2\rho^{(1)}c_rk\coth kh_1-\rho^{(1)}\alpha} \end{aligned} \tag{1-39}$$

From (1-34) $\frac{1}{F_{2r}}<0$ at $k\to0$, but from (1-37) $\frac{1}{F_{2r}}>0$ at sufficiently large value of k.

c_r in (1-39) is given as a progressive wave,

$$c_r=\frac{\rho^{(1)}\alpha+\sqrt{(\rho^{(1)}\alpha)^2+4(\rho^{(2)}k\coth kh_2+\rho^{(1)}k\coth kh_1)(\rho^{(2)}-\rho^{(1)})g}}{2(\rho^{(2)}k\coth kh_2+\rho^{(1)}k\coth kh_1)} \tag{1-40}$$

The important point of this treatment is the estimation of F_{2i}. From (1-34), at $k\to0$, it is clear that

$$F_{2i}\simeq-\frac{U_c^{(1)\prime\prime}}{U_c^{(1)\prime3}}\pi \tag{1-41}$$

We rather consider the case in which kh_1 is moderately large. In the case in which kh_1 is finite, if y_{c_r}/h_1 is small, the approximate value of F_{2i} is obtained as follows by making use of (1-26).

$$\begin{aligned} F_{2i}\simeq&-\frac{U_c^{(1)\prime\prime}}{U_c^{(1)\prime3}}\pi-\frac{U_c^{(1)\prime\prime}}{U_c^{(1)\prime3}}\pi\frac{k^2h_1^2}{6}+\frac{U_c^{(1)\prime\prime2}}{U_c^{(1)\prime4}}\pi h_1\frac{k^2h_1^2}{9}-\frac{U_c^{(1)\prime\prime}}{U_c^{(1)\prime3}}\pi\frac{k^4h_1^4}{120} \\ &+\frac{U_c^{(1)\prime\prime2}}{U_c^{(1)\prime4}}\pi h_1\frac{13}{2700}k^4h_1^4+\frac{U_c^{(1)\prime\prime3}}{U_c^{(1)\prime5}}\pi h_1^2\frac{19}{4536}k^4h_1^4-\frac{U_c^{(1)\prime\prime4}}{U_c^{(1)\prime6}}\pi h_1^3\frac{1}{441}k^4h_1^4+\cdots\cdot \end{aligned} \tag{1-42}$$

The estimation of F_{2i} is not easy from (1-42), when kh_1 is large, and we use the following procedure.

When the flow of the upper layer is shown by $U^{(1)}=\alpha y$, and the flow of the lower layer does not exist, the mechanical energy of interfacial wave can be expressed by (here terms of order of c_i^2 is disregarded at $c=c_r+ic_i$),

kinetic energy

$$\begin{aligned} E_k=&\frac{1}{4}A^{(0)2}g(\rho^{(2)}-\rho^{(1)})+\frac{1}{8}\frac{(\rho^{(1)}\alpha)^2A^{(0)2}}{k(\rho^{(2)}+\rho^{(1)})} \\ &+\frac{1}{4}\rho^{(1)}\alpha A^{(0)2}\sqrt{\frac{(\rho^{(1)}\alpha)^2}{4k^2(\rho^{(1)}+\rho^{(2)})^2}+\frac{(\rho^{(2)}-\rho^{(1)})}{k(\rho^{(1)}+\rho^{(2)})}g} \end{aligned} \tag{1-43}$$

potential energy

$$E_p=\frac{1}{4}A^{(0)2}g(\rho^{(2)}-\rho^{(1)}) \tag{1-44}$$

In (1–43) and (1–44), both kh_1 and kh_2 are treated to have large positive values. Now $A^{(0)}$ is assumed to increase with time, and is shown by $A^{(0)}=A_0^{(0)}e^{kc_it}$. This amplification of interfacial wave can be caused by the mechanical property of the general flow, which includes the moderate effect of the second derivative of the velocity profile. This means the existence of F_{2i} to determine $\frac{d}{dt}(E_k+E_p)$ in (1–43) and (1–44).

In consideration of (1–40),

$$\frac{d}{dt}(E_p+E_k)=kc_iA^{(0)2}c_r\left(\rho^{(2)}c_rk+\rho^{(1)}c_rk-\frac{1}{2}\rho^{(1)}\alpha\right) \tag{1–45}$$

The treatment of development of wind waves by J.W. Miles (1960) can be also applied to the development of interfacial waves in the present case, and the essential point of his method is

$$\left.\begin{aligned}&\frac{d}{dt}(E_p+E_k)=c_r\tau\\&\tau=-\frac{\pi}{2}\rho^{(1)}k|\varphi_c^{(1)}|^2\frac{U_c^{(1)\prime\prime}}{U_c^{(1)\prime}}\end{aligned}\right\} \tag{1–46}$$

Using (1–28), $\varphi^{(1)}(y_c)$ can be shown as follows at the condition of $c_i\to+0$,

$$\begin{aligned}\varphi^{(1)}(y_c)\simeq A^{(0)}k^2\left(-\frac{1}{3}y_{c_r}^3U_c^{(1)\prime}+\frac{1}{4}y_{c_r}^4U_c^{(1)\prime\prime}\right)+\frac{A^{(0)}}{F_2}\frac{1}{U_c^{(1)\prime}}\\+\frac{A^{(0)}}{F_2}k^2\left(\frac{1}{6}\frac{y_{c_r}^2}{U_c^{(1)\prime}}+\frac{1}{36}\frac{U_c^{(1)\prime\prime}}{U_c^{(1)\prime 2}}y_{c_r}^3\right)+\cdots\end{aligned} \tag{1–47}$$

The expression of c_i introduced by (1–45) and (1–46) is

$$c_i=-\frac{\pi}{2}\rho^{(1)}\frac{|\varphi_c^{(1)}|^2}{A^{(0)2}\left(\rho^{(2)}c_rk+\rho^{(1)}c_rk-\frac{1}{2}\rho^{(1)}\alpha\right)}\frac{U_c^{(1)\prime\prime}}{U_c^{(1)\prime}} \tag{1–48}$$

At the present procedure, c_i of (1–48) should be equal to c_i of (1–39) with the condition that kh_1 and kh_2 are very large. That is

$$F_{2i}c_r^2(c_rk-\alpha)^2=-\pi\frac{|\varphi_c^{(1)}|^2}{A^{(0)2}}\frac{U_c^{(1)\prime\prime}}{U_c^{(1)\prime}} \tag{1–49}$$

In (1–49) F_{2i} and $|\varphi_c^{(1)}|^2$ are expressed by (1–42) and (1–47) respectively. With the present simplified form of $\frac{1}{F_{2r}}\simeq c_r^2k-\alpha c_r$ and the conditions of $h_1\gg y_{c_r}$ and of $F_{2r}\gg F_{2i}$,

$$\left.\begin{aligned}&F_{2i}\simeq-\pi\frac{U_c^{(1)\prime\prime}}{U_c^{(1)\prime 3}}\\&|\varphi_c^{(1)}|^2\simeq\frac{A^{(0)2}}{F_{2r}^2}\frac{1}{U_c^{(1)\prime 2}}\end{aligned}\right\} \tag{1–50}$$

Here the important condition is $h_1 \gg y_{c_r}$. If this is established, F_{2i} of (1–42) can be approximated by the first term even in the case in which kh_1 is very large. The expressions of c_r and c_i are approximately determined by this way, and this determination should be examined from the aspect of the momentum increase of interfacial wave by the second relation of (1–46). This is done as follows.

The momentum transport of interfacial wave in the upper layer is computed noticing the existence of $U^{(1)}=\alpha y$.

$$M^{(1)}=\frac{1}{2}\rho^{(1)}A^{(0)2}kc_r-\frac{1}{4}\rho^{(1)}\alpha A^{(0)2} \tag{1-51}$$

Using the amplification $A^{(0)}=A_0^{(0)}e^{kc_it}$,

$$\frac{dM^{(1)}}{dt}=\rho^{(1)}k^2c_ic_rA^{(0)2}-\frac{1}{2}\rho^{(1)}\alpha kc_iA^{(0)2} \tag{1-52}$$

In the same way for the lower layer,

$$M^{(2)}=\frac{1}{2}\rho^{(2)}A^{(0)2}kc_r \tag{1-53}$$

$$\frac{dM^{(2)}}{dt}=\rho^{(2)}k^2c_ic_rA^{(0)2} \tag{1-54}$$

and so

$$\frac{dM}{dt}=\frac{d}{dt}(M^{(1)}+M^{(2)})=(\rho^{(1)}+\rho^{(2)})k^2c_ic_rA^{(0)2}-\frac{1}{2}\rho^{(1)}\alpha kc_iA^{(0)2} \tag{1-55}$$

c_i of (1–55) is transformed by making use of (1–48) and (1–50), and then the expression of τ in the second relation of (1–46) is used. The result is

$$\frac{dM}{dt}=\tau \tag{1-56}$$

This means the momentum relation is also satisfied within the present approximation.

Two examples of numerical estimation of amplification factor kc_i are given by the above-mentioned method. We consider the case of idealized estuarial model, and so $\rho^{(1)}=1.000$ and $\rho^{(2)}=1.020$ are used. Example–1 shows a case of a small scale model test of the rivermouth. Conditions for flows are given by

$$h_2\to\infty,\ h_1=10\,\text{cm},\quad U^{(1)}=-0.12y^2+2.4y\ \text{cm/sec} \tag{1-57}$$

and so $U^{(1-s)}=12\,\text{cm/sec}$. Using the observed result that the wave length of interfacial wave in such model is usually $1\sim10$ cm, the results of computation in the present procedure are expressed as follows. (Table–1)

In Table–1 the condition of $h_1\gg y_{c_r}$ is almost satisfied, and stillmore y_{c_r} is far greater than the expected thickness of so-called "wall friction layer" of T.B. Banjamin (1959), which is active in the vicinity of the interface. Values of $(c_r)_{\text{K.H.}}$ and $(c_i)_{\text{K.H.}}$ are given by the computation of the Kelvin-Helmholtz instability, when the uniform flow is assumed in the upper layer with the same total discharge as $U^{(1)}$ of (1–57). $(c_i)_{\text{K.H.}}$ is far greater than c_i of present proceure. This may be

Table 1

L (cm)	k	y_{o_r} (cm)	$U_c^{(1)'}$	c_r (cm/sec)	c_i (cm/sec)	kc_i	$(c_r)_{K.H.}$ (cm/sec)	$(c_i)_{K.H.}$ (cm/sec)
2	3.14	0.820	2.20	1.88	0.210	0.661	3.96	3.59
3	2.09	1.03	2.15	2.34	0.251	0.527	3.96	3.37
4	1.57	1.22	2.10	2.75	0.284	0.446	3.96	3.13
5	1.25	1.39	2.06	3.11	0.311	0.391	3.96	2.87
6	1.04	1.55	2.02	3.44	0.334	0.349	3.96	2.59
7	0.897	1.71	1.98	3.75	0.354	0.318	3.96	2.27
8	0.785	1.86	1.95	4.05	0.371	0.291	3.96	1.90

used in exceptional cases of strong jet like flow of the upper layer.

Example-2 is for field observation, and the conditions of flow are

$$h_2 \to \infty,\ h_1 = 200\,\text{cm},\ U^{(1)} = -0.00225y^2 + 0.9y\ \text{cm/sec} \tag{1-58}$$

and so $U^{(1-s)} = 90$ cm/sec. The length of interfacial waves considered and the result of computation are shown in Table-2. The condition of $h_1 \gg y_{o_r}$ may be also established in Table-2.

Table 2

L (cm)	k	y_{o_r} (cm)	$U_c^{(1)'}$	c_r (cm/sec)	c_i (cm/sec)	kc_i	$(c_r)_{K.H.}$ (cm/sec)	$(c_i)_{K.H.}$ (cm/sec)
10	0.628	4.68	0.878	4.17	0.148	0.0931	29.7	29.7
30	0.209	8.57	0.861	7.55	0.209	0.0439	29.7	29.2
50	0.125	11.48	0.848	10.04	0.231	0.0290	29.7	28.6
100	0.0628	17.44	0.821	15.01	0.211	0.0132	29.7	27.3
150	0.0418	22.57	0.798	19.17	0.159	0.00668	29.7	25.8
200	0.0314	27.35	0.776	22.93	0.0996	0.00313	29.7	24.3

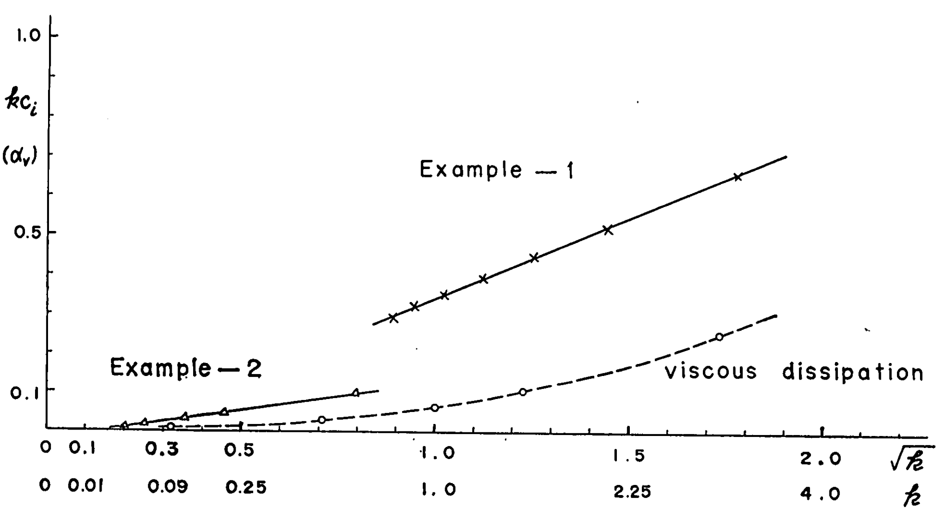

Fig. 2. Values of kc_i and α_ν against k

As values of y_{c_r} of these tables are situated at the far outside of wall friction layer of the interface, the effect of viscosity for the amplification of interfacial waves may be very small. Accordingly we only consider the attenuation effect of viscosity. The approximate mechanism of viscous attenuation in section 1.2 and 1.4 of Part I will be applied to the present problem. We put the amplitude of interfacial wave at the interface $A_0^{(0)}e^{-\alpha_\nu t}$ considering only the viscous attenuation, and from (1–34) (or more accurate from (1–21)) of Part I,

$$\left.\begin{aligned} \alpha_\nu &\doteqdot \frac{(-n_0)^{1/2}}{2\sqrt{2}} k\sqrt{\nu} \\ n_0 &= -\sqrt{\frac{gk(\rho^{(2)}-\rho^{(1)})}{\rho^{(2)}+\rho^{(1)}}} \end{aligned}\right\} \tag{1-59}$$

The relation between k and α_ν ($\nu=0.01$ approximately) is

k	3	1.5	1.0	0.5	0.1
α_ν	0.248	0.103	0.0612	0.0262	0.00353

Values of kc_i and α_ν against k in the above-mentioned two examples are shown in Fig. 2. Because $kc_i-\alpha_\nu$ is positive (assuming a linear combination of them), interfacial waves may be amplified. The expression $\dfrac{1}{kc_i-\alpha_\nu}$ may be a typical time duration necessary for the amplification, and this is not so large. From Fig. 1 $\dfrac{1}{kc_i-\alpha_\nu}$ may be several seconds in the model test. On the other side the wave celerity c_r is several centimetres per second. Interfacial waves in such model experiment will be largely amplified at the relatively short running distance (may be less than 1 m), and will be probable to become unstable and to begin the turbulent mixing.

1–2. Appendix—The decrease of wave celerity by the existence of mixed layer at the interface

When the mixed layer is formed at the interface, the characteristics of internal wave (the name of interfacial wave is not proper in this case) becomes different from the interfacial wave. The most remarkable one is the decrease of wave celerity of the principal mode of the internal wave in response to the thickness of the mixed layer.

A three layer model is considered. The general flow is not included. The model in still condition is as follows. (Fig. 3)

$$\begin{array}{lll} \text{II-layer} & -h_2\leq y\leq 0 & \rho=\rho^{(2)}=\text{const.} \\ \text{III-layer} & 0\leq y\leq y_1 & \rho=\rho^{(2)}e^{-\alpha y}\doteqdot\rho^{(2)}(1-\alpha y) \\ \text{I-layer} & y_1\leq y\leq h_1 & \rho=\rho^{(2)}(1-\alpha y_1)=\rho^{(1)}=\text{const} \end{array} \tag{1-60}$$

We are mainly treating the density difference of salt and fresh water. Therefore αy_1 may be about 0.02, and the approximation of $e^{-\alpha y}\doteqdot 1-\alpha y$ is sufficiently reliable.

The analysis is of linear perturbed waves in inviscid condition, and at $y=-h_2$ and h_1 the vertical movement of liquid is neglected. This is correct for the fixed horizontal boundary at the bottom, and for the free surface at $y=h_1$ this is an allowable approximation. Stream functions of three layers are given by

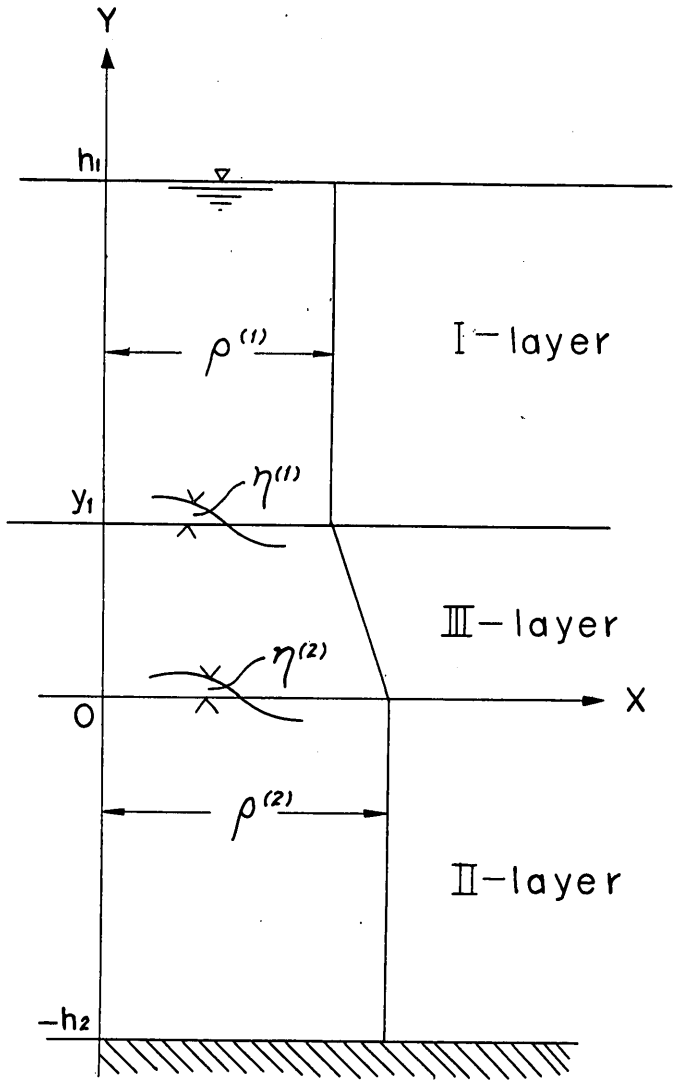

Fig. 3. Schematic representation of assumed model

$$\begin{aligned} &\text{II-layer} && u^{(2)}=-\psi_y^{(2)}, \quad v^{(2)}=\psi_x^{(2)} \\ &\text{III-layer} && u^{(3)}=-\psi_y^{(3)}, \quad v^{(3)}=\psi_x^{(3)} \qquad (1\text{-}61) \\ &\text{I-layer} && u^{(1)}=-\psi_y^{(1)}, \quad v^{(1)}=\psi_x^{(1)} \end{aligned}$$

The vertical displacement of equi-density plane at $y=0$ is shown by $A^{(2)}e^{i(kx-\sigma t)}$, at $y=y_1$ it is shown by $A^{(1)}e^{i(kx-\sigma t)}$.

There is no difficulty for expressions of $\psi^{(1)}$ and $\psi^{(2)}$. For $\psi^{(3)}$ we may compute in the refrence of H. Lamb (1932, pp. 378–380). In consideration of conjoining of the vertical velocity at $y=0$ and $y=y_1$ and of the disappearance of the vertical velocity at $y=-h_2$ and $y=h_1$, stream functions are shown by

$$\text{II-layer} \qquad \psi^{(2)}=\frac{-\sigma A^{(2)}}{k\sinh kh_2}\sinh k(y+h_2)e^{i(kx-\sigma t)} \qquad (1\text{-}61)$$

I-layer $$\phi^{(1)}=\frac{-\sigma A^{(2)}e^{(\alpha/2)y_1}}{k\sinh k(y_1-h_1)}\frac{\cos(\beta y_1+\varepsilon)}{\cos\varepsilon}\sinh k(y-h_1)e^{i(kx-\sigma t)} \tag{1-63}$$

III-layer $$\phi^{(3)}=-\frac{\sigma A^{(2)}}{k}e^{(\alpha/2)y}\frac{\cos(\beta y+\varepsilon)}{\cos\varepsilon}e^{i(kx-\sigma t)} \tag{1-64}$$

Here

$$A^{(1)}=A^{(2)}e^{(\alpha/2)y_1}\frac{\cos(\beta y_1+\varepsilon)}{\cos\varepsilon} \tag{1-65}$$

$$\beta=k\sqrt{\left(\frac{g\alpha}{\sigma^2}-1\right)-\frac{1}{4}\frac{\alpha^2}{k^2}}=kX \tag{1-66}$$

From the conjunction of pessure at the flexible boundary of $y=0$,

$$k\coth kh_2=\frac{\alpha}{2}-\beta\tan\varepsilon \tag{1-67}$$

From the conjunction of pressure at $y=y_1$,

$$\frac{\alpha}{2}\cos\beta y_1-\beta\sin\beta y_1-k\cos\beta y_1\coth k(y_1-h_1)$$

$$=\left\{\frac{\alpha}{2}\sin\beta y_1+\beta\cos\beta y_1-k\sin\beta y_1\coth k(y_1-h_1)\right\}\tan\varepsilon \tag{1-68}$$

The general form of the characteristic equation is obtained from (1-67) and (1-68) by the elimination of $\tan\varepsilon$. But the discussion of the general form is complicated, and we limit the treatment into following two cases; (i) the case in which kh_2 and $k(h_1-y_1)$ are very large. (ii) the case in which the wave length of internal wave is very large compared with the total depth h_1+h_2.

Firstly we treat the case in which $\coth k(y_1-h_1)\to-1$ and $\coth kh_2\to1$ are consistent. The characteristic equation is

$$(ky_1X)^2\tan ky_1X+\frac{(\alpha y_1)^2}{4}\tan ky_1X-2(ky_1)^2X-(ky_1)^2\tan ky_1X=0 \tag{1-69}$$

We can use the Boussinesq approximation to (1-69) referring the condition $\alpha y_1\simeq 0.02$. The term $\frac{(\alpha y_1)^2}{4}\tan ky_1X$ is neglected in (1-69), and in (1-66) a simple form $\sqrt{\frac{g\alpha}{\sigma^2}-1}\doteqdot X$ is used. The solution of (1-69) contains principal harmonic and also biharmonics. For the reference the solution of (1-69) obtained as abovementioned is shown in Table 3. We see that the phase of $u^{(3)}$ of biharmonics varies significantly between $y=0$ and $y=y_1$, and this is of course expected from the simpler example by H. Lamb (1932). These biharmonics, which satisfy strictly idealized conditions of (1-60), are not so important in practical meaning, and we only remark the property of principal harmonic.

If we put the eigenvalue of the principal harmonic as

$$\sigma=\left\{\frac{gk\alpha y_1}{2}\right\}^{1/2}(1-\delta) \tag{1-70}$$

Table 3

ky_1	$X_{\text{principal}}$	$X_{\text{the second}}$	$X_{\text{the third}}$	$X_{\text{the fourth}}$
0.1	4.434			
0.2	3.110			
0.5	1.919			
1.0	1.306	3.668	6.112	
1.5	1.028	2.586	4.481	6.487
2.0	0.860	2.029	3.425	4.913
2.5	0.745	1.695	2.789	3.967
3.0	0.659	1.450	2.362	3.337
3.5	0.592	1.277	2.054	2.883
4.0	0.539	1.144	1.822	2.540
4.5	0.494	1.039	1.640	2.278
5.0	0.457	0.954	1.492	2.065

δ means the decrease of the celerity of the internal wave in the present case to the interfacial wave. In Table 4 values of δ of (1-70) and ε of (1-64) are recorded against ky_1. From Table 4 the decrease of wave celerity is obvious with the incerase of the thickness of the mixed layer, and this tendency is remarkable when the thickness of the layer is comparable with the wave length. ε in Table 4 connected with X principal in Table 3 indicates that horizontal particle velocity at both sides of the mixed layer is equal and opposite. The sign changes in the midway of the mixed layer.

Table 4

ky_1	0.1	0.2	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
δ	0.017	0.034	0.076	0.141	0.195	0.242	0.283	0.318	0.349	0.377	0.402	0.424
ε	−0.222	−0.311	−0.480	−0.653	−0.771	−0.860	−0.931	−0.988	−1.036	−1.076	−1.112	−1.142

Secondly we treat the case in which the wave length of the internal wave is sufficiently greater than the total depth h_1+h_2, and so approximations of $\coth k(y_1-h_1)\doteqdot\frac{1}{k(y_1-h_1)}$ and $\coth kh_2\doteqdot\frac{1}{kh_2}$ are used.

The characteristic equation in this case is

$$-(ky_1X)^2\tan ky_1X+(ky_1X)\frac{y_1}{h_1-y_1}+(ky_1X)\frac{y_1}{h_2}+\frac{y_1}{h_2}\frac{y_1}{h_1-y_1}\tan ky_1X$$

$$=\left(\frac{\alpha y_1}{2}\right)^2\tan ky_1X+\frac{\alpha y_1}{2}\frac{y_1}{h_1-y_1}\tan ky_1X-\frac{\alpha y_1}{2}\frac{y_1}{h_2}\tan ky_1X \qquad (1\text{-}71)$$

If y_1 is very small in (1-71), we have

$$c^2=\frac{g\alpha y_1}{\left(1-\frac{\alpha y_1}{2}\right)\frac{1}{h_1-y_1}+\left(1+\frac{\alpha y_1}{2}\right)\frac{1}{h_2}+\frac{y_1}{h_1-y_1}\frac{1}{h_2}} \qquad (1\text{-}72)$$

Table 5

case		(0.1, 0.1)	(0.2, 0.2)	(0.3, 0.3)
c/c_0	$ky_1=0.01$	0.988	0.974	0.960
	$ky_1=0.001$	0.989	0.974	0.960

and on the other hand in the case of two-layer liquid we have

$$c_0^2=\frac{(\rho^{(2)}-\rho^{(1)})g}{\dfrac{\rho^{(1)}}{h_1}+\dfrac{\rho^{(2)}}{h_2}} \tag{1-73}$$

(in (1–73) h_1 is the depth of the upper layer, and h_2 is the depth of the lower layer). (1–72) is connected to (1–73) at $y_1 \to 0$, and using the premise that αy_1 is constant, the wave celerity has the tendency to decrease with the increase of y_1. But y_1 in (1–72) is too small compared with h_1 and h_2, and the solution for practical use is obtained by the numerical trial of (1–71).

Putting $\alpha y_1=0.02$, we seek solutions of (1–71) of three cases in which $\left(\dfrac{y_1}{h_2}, \dfrac{y_1}{h_1-y_1}\right)$ is (0.1, 0.1), (0.2, 0.2) and (0.3, 0.3) respectively, and these soulutions are compared with solutions of (1–73) in which the interface of two fluids is taken at $y=\dfrac{y_1}{2}$ (the middle point of the mixed layer). Here $\rho^{(2)}=1.020$ and $\rho^{(1)}=1.000$ are used in (1–73). The result is shown in Table 5. This table shows that the decrease of the wave celerity is not significant in the range of the assumed thickness of the mixed layer. If the mixed layer is enlarged to cover the total depth (from $y=-h_2$ to $y=h_1$), we can use the estimation of H. Lamb (1932, pp. 378–380).

$$c \fallingdotseq \sqrt{gh}\,\frac{\left(\dfrac{\Delta\rho}{\rho}\right)^{1/2}}{2} \tag{1-74}$$

Here $\Delta\rho$ is the density difference taken at the lower and upper boundary, and h means the total depth. On the contrary, if we use $h_1=h_2=\dfrac{h}{2}$ in (1–73), we have

$$c_0 \fallingdotseq \sqrt{gh}\,\frac{\left(\dfrac{\Delta\rho}{\rho}\right)^{1/2}}{2} \tag{1-75}$$

Here $\Delta\rho=\rho^{(2)}-\rho^{(1)}$. Thus the extreme case may be given by $c/c_0=2/\pi$, setting virtually $\Delta\rho$ of (1–74) and (1–75) as the same.

1–3. An stability analysis of internal waves by shear flow

A three-layer model with a mixed layer is examined. A shear flow is monotonous, and the curvature of the velocity profile is negligible. This type of stability problem relates practically to the internal wave at the rivermouth where we can find the superposition of fresh and salt water with the intermediate mixed layer. It was already reported by L. N. Howard (1961) that the internal wave

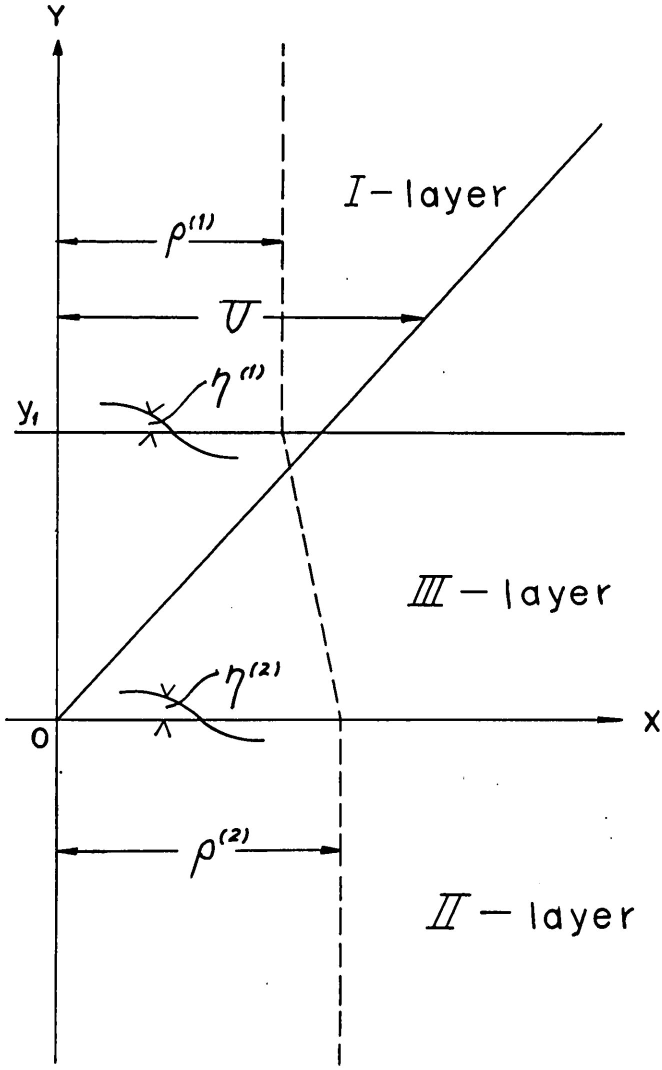

Fig. 4. Schematic representation of assumed model

which is not forced externally is always stable for the Richardson number greater than $\frac{1}{4}$. We treat here the case of $0<R_i<\frac{1}{4}$.

The present model is shown as follows (Fig. 4);

	layer	density	flow
II-layer	$-\infty\leq y\leq 0$	$\bar{\rho}=\rho^{(2)}=\text{const.}$	$U=U^{(2)}=0$
III-layer	$0\leq y\leq y_1$	$\bar{\rho}=\rho^{(3)}=\rho^{(2)}e^{-\alpha y}=\rho^{(2)}(1-\alpha y)$	$U=U^{(3)}=ay$
I-layer	$y_1\leq y<\infty$	$\bar{\rho}=\rho^{(1)}=\rho^{(2)}(1-\alpha y_1)$	$U=U^{(1)}=ay$

(1-76)

Here a is constant (real and positive). αy_1 is small, and is about 0.02 in practice.

The feature of this model is that U linearly increases in I-layer, (for instance in fresh water layer). A model, in which U=const. in I-layer, and U increases linearly in III-layer, is already examined by J.W. Miles and L.N. Howard (1964). In (1–76) we elongated II-layer to $-\infty$, and also I-layer to $+\infty$. But in practice this means that the thickness of II- and I-layer is two or three times or more greater than the length of internal waves.

We use the expression of stream function of (1-3).

In II- and I-layer

$$\varphi_{yy}^{(i)}-k^2\varphi^{(i)}=0 \qquad (i=2,\ 1) \tag{1-77}$$

In III-layer

$$\varphi_{yy}^{(3)}-\alpha\varphi_y^{(3)}-\left\{k^2-\alpha g\frac{1}{(c_r-U+ic_i)^2}+\frac{\alpha a}{c_r-U+ic_i}\right\}\varphi^{(3)}=0 \tag{1-78}$$

In II- and I-layer, φ is given by $\varphi\sim e^{\pm ky}$. The vertical displacement of equi-density plane from the still condition at $y=0$ is expressed by $\eta^{(2)}$, and the same quantity at $y=y_1$ is given by $\eta^{(1)}$. These are periodic and expressed by

$$\left.\begin{aligned}\eta^{(2)}&=A_0^{(2)}e^{ik(x-ct)}\\ \eta^{(1)}&=A_0^{(1)}e^{ik(x-ct)}\end{aligned}\right\} \tag{1-79}$$

Corresponding properties of waves at II- and I-layer are given by;

in II-layer

$$\left.\begin{aligned}\phi^{(2)}&=-cA_0^{(2)}e^{ky}e^{ik(x-ct)}\\ u^{(2)}&=kcA_0^{(2)}e^{ky}e^{ik(x-ct)}\\ v^{(2)}&=-ikcA_0^{(2)}e^{ky}e^{ik(x-ct)}\\ p^{(2)}&=-\rho^{(2)}gy+\rho^{(2)}kc^2A_0^{(2)}e^{ky}e^{ik(x-ct)}\end{aligned}\right\} \tag{1-80}$$

in I-layer

$$\left.\begin{aligned}\phi^{(1)}&=A_0^{(1)}(ay_1-c)e^{-k(y-y_1)}e^{ik(x-ct)}\\ u^{(1)}&=A_0^{(1)}k(ay_1-c)e^{-k(y-y_1)}e^{ik(x-ct)}\\ v^{(1)}&=ikA_0^{(1)}(ay_1-c)e^{-k(y-y_1)}e^{ik(x-ct)}\\ p^{(1)}&=\frac{1}{\alpha}(\rho^{(1)}-\rho^{(2)})g-\rho^{(1)}g(y-y_1)+\rho^{(1)}A_0^{(1)}\\ &\quad\times(ay_1-c)e^{-k(y-y_1)}(ck-ayk-a)e^{ik(x-ct)}\end{aligned}\right\} \tag{1-81}$$

In III-layer, putting $\varphi=e^{(\alpha/2)y}X$ in (1-78)

$$X_{yy}+\left\{-k^2-\alpha\frac{a}{c-U}+g\alpha\frac{1}{(c-U)^2}-\frac{\alpha^2}{4}\right\}X=0 \tag{1-82}$$

As c is complex, if $U(y_c)=c_r+ic_i$, by making use of $U=ay$, we have

$$y_{o_r}=\frac{c_r}{a}, \quad y_{o_i}=\frac{c_i}{a} \tag{1-83}$$

(1-82) is simplified by the Boussinesq approximation.

$$X_{yy}+\left\{-k^2+g\alpha\frac{1}{a^2(y-y_o)^2}\right\}X=0 \tag{1-84}$$

Therefore the solution of (1-78) is

$$\varphi=e^{(\alpha/2)y}(AX_+ + BX_-) \tag{1-85}$$

Here

$$X_+=(y-y_o)^{1/2+\delta/2}\left\{1+\frac{k^2}{4+2\delta}(y-y_o)^2+\frac{k^2}{16+4\delta}\frac{k^2}{4+2\delta}(y-y_o)^4\right.$$
$$+\frac{k^2}{36+6\delta}\frac{k^2}{16+4\delta}\frac{k^2}{4+2\delta}(y-y_o)^6+\frac{k^2}{64+8\delta}\frac{k^2}{36+6\delta}\frac{k^2}{16+4\delta}$$
$$\left.\times\frac{k^2}{4+2\delta}(y-y_o)^8+\cdots\cdots\right\}$$

X_- is obtained from X_+ by $\delta\to-\delta$.

δ is given by

$$\delta=\sqrt{1-\frac{4g\alpha}{a^2}} \qquad \left(\frac{g\alpha}{a^2}=R_i\right) \tag{1-86}$$

δ is real and positive in this problem, and it satisfies $1>\delta>0$. The modified Bessel function can be conveniently used to (1-85), but in the reference of the convenience of algebraic computations for the numerical determination of c_i, X_+ and X_- are used in this treatment.

Using (1-85) characteristics of waves in III-layer are shown by

$$\left.\begin{aligned}
&\phi^{(3)}=e^{(\alpha/2)y}(AX_+ + BX_-)e^{ik(x-ct)}\\
&u^{(3)}=\left\{-\frac{\alpha}{2}e^{(\alpha/2)y}(AX_+ + BX_-)-e^{(\alpha/2)y}(AX'_+ + BX'_-)\right\}\\
&\qquad\times e^{ik(x-ct)}\\
&v^{(3)}=ike^{(\alpha/2)y}(AX_+ + BX_-)e^{ik(x-ct)}\\
&p^{(3)}=\frac{g}{\alpha}(\rho^{(2)}e^{-\alpha y}-\rho^{(2)})-\rho^{(2)}e^{-(\alpha/2)y}\left[\left\{\frac{\alpha}{2}(AX_+ + BX_-)\right.\right.\\
&\qquad\left.\left.+(AX'_+ + BX'_-)\right\}(c-ay)+a(AX_+ + BX_-)\right]e^{ik(x-ct)}
\end{aligned}\right\} \tag{1-87}$$

X'_+ and X'_- mean the differential derivatives of X_+ and X_- with respect to y respectively.

The characteristic equation are introduced by (1-80), (1-81) and (1-87). Necessary boundary conditions are

$$v^{(3)}=v^{(2)}, \quad p^{(3)}=p^{(2)} \quad \text{at} \quad y=0$$
$$v^{(3)}=v^{(1)}, \quad u^{(3)}=u^{(1)} \quad \text{at} \quad y=y_1 \tag{1-88}$$

It is noted that the discontinuity of density does not appear at $y=0$ and $y=y_1$.

The characteristic equation obtained is expressed by

$$\left[\left(kc-\frac{\alpha c}{2}-a\right)X_{+<0>}-cX'_{+<0>}\right]\left[\left(k+\frac{\alpha}{2}\right)X_{-<y_1>}+X'_{-<y_1>}\right]$$
$$-\left[\left(kc-\frac{\alpha c}{2}-a\right)X_{-<0>}-cX'_{-<0>}\right]\left[\left(k+\frac{\alpha}{2}\right)X_{+<y_1>}+X'_{+<y_1>}\right]=0 \tag{1-89}$$

$$\left.\begin{aligned} A&=\frac{\left(ky_c-\frac{\alpha}{2}y_c-1\right)X_{-<0>}-y_cX'_{-<0>}}{X_{+<0>}X'_{-<0>}-X_{-<0>}X'_{+<0>}}aA_0^{(2)} \\ B&=-\frac{\left(ky_c-\frac{\alpha}{2}y_c-1\right)X_{+<0>}-y_cX'_{+<0>}}{X_{+<0>}X'_{-<0>}-X_{-<0>}X'_{+<0>}}aA_0^{(2)} \end{aligned}\right\} \tag{1-90}$$

In notations $<0>$ means $y=0$, and $<y_1>$ means $y=y_1$. Three conditions as follows are used to solve (1-89).

(i) y_{c_r}, which statisfies $U(y_{c_r})=c_r$, situates between $y=0$ and $y=y_1$. ($y_1>y_{c_r}>0$)

(ii) The method of the singular neutral mode for $c_i \rightarrow +0$ pointed out by J.W. Miles (1961, 1963) is used. For instance $(-y_c)^{1/2+\delta/2}=e^{-\pi i(1/2+\delta/2)}y_c^{1/2+\delta/2}$ in the present study.

(iii) A case, in which $/y_{c_i}/$ is sufficiently smaller than both $y_1-y_{c_r}$ and y_{c_r}, is taken into consideration, and so

$$\left.\begin{aligned} &(y_1-y_{c_r}-iy_{c_i})^{1/2\pm\delta/2} \fallingdotseq (y_1-y_{c_i})^{1/2\pm\delta/2}-i\left(\frac{1}{2}\pm\frac{\delta}{2}\right)y_{c_i}(y_1-y_{c_r})^{\pm\delta/2-1/2} \\ &(y_{c_r}+iy_{c_i})^{1/2\pm\delta/2} \fallingdotseq y_{c_r}{}^{1/2\pm\delta/2}+i\left(\frac{1}{2}\pm\frac{\delta}{2}\right)y_{c_i}y_{c_r}{}^{\pm\delta/2-1/2} \end{aligned}\right\} \tag{1-91}$$

Using these conditions, next relations are obtained subsequently.

$$X_{+<y_1>}=(X_{+<y_1>})_r-iy_{c_i}(X_{+<y_1>})_{i1} \tag{1-92}$$

$$X'_{+<y_1>}=(X'_{+<y_1>})_r-iy_{c_i}(X'_{+<y_1>})_{i1} \tag{1-93}$$

$$X_{+<0>}=-ie^{-i(\delta/2)\pi}(X_{+<0>})_{r1}+e^{-i(\delta/2)\pi}y_{c_i}(X_{+<0>})_{r2} \tag{1-94}$$

$$X'_{+<0>}=ie^{-i(\delta/2)\pi}(X'_{+<0>})_{r1}-e^{-i(\delta/2)\pi}y_{c_i}(X'_{+<0>})_{r2} \tag{1-95}$$

Here

$$(X_{+<y_1>})_r=(y_1-y_{c_r})^{1/2+\delta/2}+\frac{k^2}{4+2\delta}(y_1-y_{c_r})^{5/2+\delta/2}+\frac{k^2}{16+4\delta}\frac{k^2}{4+2\delta}$$
$$\times(y_1-y_{c_r})^{9/2+\delta/2}+\frac{k^2}{36+6\delta}\frac{k^2}{16+4\delta}\frac{k^2}{4+2\delta}(y_1-y_{c_r})^{13/2+\delta/2}+\cdots\cdots \tag{1-96}$$

$$(X_{+<y_1>})_{i1}=(X'_{+<y_1>})_r=\left(\frac{1}{2}+\frac{\delta}{2}\right)(y_1-y_{o_r})^{-1/2+\delta/2}+\frac{k^2}{4+2\delta}\left(\frac{5}{2}+\frac{\delta}{2}\right)$$
$$\times(y_1-y_{o_r})^{3/2+\delta/2}+\frac{k^2}{16+4\delta}\frac{k^2}{4+2\delta}\left(\frac{9}{2}+\frac{\delta}{2}\right)(y_1-y_{o_r})^{7/2+\delta/2}$$
$$+\frac{k^2}{36+6\delta}\frac{k^2}{16+4\delta}\frac{k^2}{4+2\delta}\left(\frac{13}{2}+\frac{\delta}{2}\right)(y_1-y_{o_r})^{11/2+\delta/2}+ \qquad (1\text{-}97)$$

$$(X'_{+<y_1>})_{i1}=\left(\frac{1}{2}+\frac{\delta}{2}\right)\left(\frac{\delta}{2}-\frac{1}{2}\right)(y_1-y_{o_r})^{\delta/2-3/2}+\left(\frac{5}{2}+\frac{\delta}{2}\right)\left(\frac{3}{2}+\frac{\delta}{2}\right)$$
$$\times\frac{k^2}{4+2\delta}(y_1-y_{o_r})^{1/2+\delta/2}+\left(\frac{9}{2}+\frac{\delta}{2}\right)\left(\frac{7}{2}+\frac{\delta}{2}\right)\frac{k^2}{16+4\delta}\frac{k^2}{4+2\delta}(y_1-y_{o_r})^{5/2+\delta/2}$$
$$+\left(\frac{13}{2}+\frac{\delta}{2}\right)\left(\frac{11}{2}+\frac{\delta}{2}\right)\frac{k^2}{36+6\delta}\frac{k^2}{16+4\delta}\frac{k^2}{4+2\delta}(y_1-y_{o_r})^{9/2+\delta/2}+\cdots\cdots \qquad (1\text{-}98)$$

$$(X_{+<0>})_{r1}=y_{c_r}^{1/2+\delta/2}\left(1+\frac{k^2}{4+2\delta}y_{c_r}^2+\frac{k^2}{16+4\delta}\frac{k^2}{4+2\delta}y_{c_r}^4\right.$$
$$\left.+\frac{k^2}{36+6\delta}\frac{k^2}{16+4\delta}\frac{k^2}{4+2\delta}y_{c_r}^6+\cdots\cdots\right) \qquad (1\text{-}99)$$

$$(X_{+<0>})_{r2}=(X'_{+<0>})_{r1}=y_{c_r}^{\delta/2-1/2}\left\{\left(\frac{1}{2}+\frac{\delta}{2}\right)+\left(\frac{5}{2}+\frac{\delta}{2}\right)\frac{k^2}{4+2\delta}y_{c_r}^2\right.$$
$$\left.+\left(\frac{9}{2}+\frac{\delta}{2}\right)\frac{k^2}{16+4\delta}\frac{k^2}{4+2\delta}y_{c_r}^4+\left(\frac{13}{2}+\frac{\delta}{2}\right)\frac{k^2}{36+6\delta}\frac{k^2}{16+4\delta}\frac{k^2}{4+2\delta}y_{c_r}^6+\cdots\cdots\right\}$$
$$(1\text{-}100)$$

$$(X'_{+<0>})_{r2}=y_{c_r}^{\delta/2-3/2}\left\{\left(\frac{1}{2}+\frac{\delta}{2}\right)\left(\frac{\delta}{2}-\frac{1}{2}\right)+\left(\frac{5}{2}+\frac{\delta}{2}\right)\left(\frac{3}{2}+\frac{\delta}{2}\right)\frac{k^2}{4+2\delta}y_{c_r}^2\right.$$
$$+\left(\frac{9}{2}+\frac{\delta}{2}\right)\left(\frac{7}{2}+\frac{\delta}{2}\right)\frac{k^2}{16+4\delta}\frac{k^2}{4+2\delta}y_{c_r}^4+\left(\frac{13}{2}+\frac{\delta}{2}\right)\left(\frac{11}{2}+\frac{\delta}{2}\right)$$
$$\left.\times\frac{k^2}{36+6\delta}\frac{k^2}{16+4\delta}\frac{k^2}{4+2\delta}y_{c_r}^6+\cdots\cdots\right\} \qquad (1\text{-}101)$$

$X_{-<y_1>}$, $X'_{-<y_1>}$, $X_{-<0>}$ and $X'_{-<0>}$ are given by the index change $\delta\to-\delta$.

By making use of the above-mentioned expressions, the characteristic equation (1-89) is transformed to the next two relations after some algebraic computations.

$$-\cot\frac{\delta}{2}\pi(M_1-M_2)=y_{o_i}(K_1+K_2) \qquad (1\text{-}102)$$

$$\tan\frac{\delta}{2}\pi(M_1+M_2)=y_{o_i}(K_1-K_2) \qquad (1\text{-}103)$$

Here

$$M_1=\left\{\left(ky_{c_r}-\frac{\alpha}{2}y_{c_r}-1\right)(X_{+<0>})_{r1}+y_{c_r}(X_{+<0>})_{r2}\right\}\left\{\left(k+\frac{\alpha}{2}\right)(X_{-<y_1>})_r+(X'_{-<y_1>})_r\right\} \tag{1-104}$$

$$M_2=\left\{\left(ky_{c_r}-\frac{\alpha}{2}y_{c_r}-1\right)(X_{-<0>})_{r1}+y_{c_r}(X_{-<0>})_{r2}\right\}\left\{\left(k+\frac{\alpha}{2}\right)(X_{+<y_1>})_r+(X'_{+<y_1>})_r\right\} \tag{1-105}$$

$$\begin{aligned}K_1=&\left\{\left(ky_{c_r}-\frac{\alpha}{2}y_{c_r}\right)(X_{+<0>})_{r2}+\left(k-\frac{\alpha}{2}\right)(X_{+<0>})_{r1}+y_{c_r}(X'_{+<0>})_{r2}\right\}\\&\times\left\{\left(k+\frac{\alpha}{2}\right)(X_{-<y_1>})_r+(X'_{-<y_1>})_r\right\}-\left\{\left(ky_{c_r}-\frac{\alpha}{2}y_{c_r}-1\right)(X_{+<0>})_{r1}\right.\\&\left.+y_{c_r}(X_{+<0>})_{r2}\right\}\left\{\left(k+\frac{\alpha}{2}\right)(X'_{-<y_1>})_r+(X'_{-<y_1>})_{i1}\right\}\end{aligned} \tag{1-106}$$

$$\begin{aligned}K_2=&\left\{\left(ky_{c_r}-\frac{\alpha}{2}y_{c_r}\right)(X_{-<0>})_{r2}+\left(k-\frac{\alpha}{2}\right)(X_{-<0>})_{r1}+y_{c_r}(X'_{-<0>})_{r2}\right\}\\&\times\left\{\left(k+\frac{\alpha}{2}\right)(X_{+<y_1>})_r+(X'_{+<y_1>})_r\right\}-\left\{\left(ky_{c_r}-\frac{\alpha}{2}y_{c_r}-1\right)(X_{-<0>})_{r1}\right.\\&\left.+y_{c_r}(X_{-<0>})_{r2}\right\}\left\{\left(k+\frac{\alpha}{2}\right)(X'_{+<y_1>})_r+(X'_{+<y_1>})_{i1}\right\}\end{aligned} \tag{1-107}$$

Firstly the case of $c_i\to 0$ (neutral mode) is considered. In this case the neutral curve is determined by

$$\left.\begin{aligned}M_1-M_2=0\\M_1+M_2=0\end{aligned}\right\} \tag{1-108}$$

and so we obtain

$$\left.\begin{aligned}&\left(ky_{c_r}-\frac{\alpha}{2}y_{c_r}-1\right)(X_{+<0>})_{r1}+y_{c_r}(X_{+<0>})_{r2}=0\\&\left(k+\frac{\alpha}{2}\right)(X_{+<y_1>})_r+(X'_{+<y_1>})_r=0\end{aligned}\right\} \tag{1-109}$$

or

$$\left.\begin{aligned}&\left(ky_{c_r}-\frac{\alpha}{2}y_{c_r}-1\right)(X_{-<0>})_{r1}+y_{c_r}(X_{-<0>})_{r2}=0\\&\left(k+\frac{\alpha}{2}\right)(X_{-<y_1>})_r+(X'_{-<y_1>})_r=0\end{aligned}\right\} \tag{1-110}$$

If small quantities are neglected, (1-109) and (1-110) are modified to

$$\left.\begin{aligned}&(ky_{c_r}-1)(X_{+<0>})_{r1}+y_{c_r}(X_{+<0>})_{r2}=0\\&k(X_{+<y_1>})_r+(X'_{+<y_1>})_r=0\end{aligned}\right\} \tag{1-109'}$$

$$\left.\begin{array}{l}(ky_{c_r}-1)(X_{-<0>})_{r1}+y_{c_r}(X_{-<0>})_{r2}=0\\ k(X_{-<y_1>})_r+(X'_{-<y_1>})_r=0\end{array}\right\} \quad (1\text{-}110')$$

By this way the neutral curve is determined by (1-109) and (1-110), or simpler by (1-109′) and (1-110′).

At this point we remark the wave induced Reynolds stress of this model. This stress is expressed by

$$-\bar{\rho}\,\overline{uv}=-\frac{1}{2}\bar{\rho}\boldsymbol{R}(\overline{uv^*})=\frac{1}{2}\bar{\rho}\boldsymbol{I}\left(\frac{\partial\varphi}{\partial z}\varphi^*\right)ke^{2kc_it} \quad (1\text{-}111)$$

With the condition of $y_{c_i}\to+0$, we compute this stress in the vicinity of y_{c_r}.

Above y_{c_r}

$$\tau_{(>y_c)}=\frac{1}{4i}\bar{\rho}(AB^*-BA^*)\delta ke^{2kc_it} \quad (1\text{-}112)$$

Below y_{c_r}

$$\tau_{(<y_c)}=\frac{-1}{4i}\bar{\rho}(AB^*e^{-i\pi\delta}-A^*Be^{i\pi\delta})\delta ke^{2kc_it} \quad (1\text{-}113)$$

Then (1-112) and (1-113) are transformed by making use of (1-90), (1-94) and (1-95) with a condition $y_{c_i}\to+0$.

$$\tau_{(>y_c)}=\frac{-1}{2}\bar{\rho}a^2A_0^{(2)2}\delta k\sin\delta\pi\cdot e^{2kc_it}$$

$$\times\frac{\left\{\left(ky_{c_r}-\frac{\alpha}{2}y_{c_r}-1\right)(X_{+<0>})_{r1}+y_{c_r}(X_{+<0>})_{r2}\right\}\left\{\left(ky_{c_r}-\frac{\alpha}{2}y_{c_r}-1\right)(X_{-<0>})_{r1}+y_{c_r}(X_{-<0>})_{r2}\right\}}{|(X_{+<0>})_{r1}(X_{-<0>})_{r2}-(X_{-<0>})_{r1}(X_{+<0>})_{r2}|^2}$$

(1-114)

$$\tau_{(<y_c)}=0 \quad (1\text{-}115)$$

Accordingly above y_c the Reynolds stress has generally finite value, and below y_c it is zero. Expressions (1-114) and (1-115) are introduced with the condition of $y_{c_i}\to+0$, and these may be applicable to very small values of y_{c_i}. If conditions of (1-109) and (1-110) for neutral curves are applied to (1-114), $\tau_{(>y_c)}$ becomes zero, and this means the wave induced Reynolds stress disappears along the neutral curve, and the finite jump of the value of Reynolds stress at y_c also disappears. In (1-114) $\tau_{(>y_c)}$ also becomes zero at $R_i\to1/4$ or at $R_i\to0$.

Next we remark to the second relation of the neutral mode in (1-109), (1-110′). According to expressions of (1-96) and (1-97), $(X_{\pm<y_1>})_r$ and $(X'_{\pm<y_1>})_r$ are always positive at the condition of $1>\delta>0$. Therefore, as k and α are positive in the present problem, the second relation of the neutral mode cannot be satisfied for any k. This means that the neutral curve cannot be drawn in the region of $\frac{1}{4}>R_i>0$. This is an important result of the present treatment.

A contrastive result is obtaind in the case in which in I-layer $(y_1\leqq y<\infty)U$ is checked by the condition $U=U^{(1)}=ay_1$. We see easily that this case is transformed to the treatment of J.W. Miles and L.N. Howard (1964) by the shift of the coördinate. In the present system of co-ordinate, the relations for neutral mode are expressed by

$$\left.\begin{aligned}(ky_{c_r}-1)(X_{+<0>})_{r1}+y_{c_r}(X_{+<0>})_{r2}=0\\(ky_{c_r}-1)(X_{+<y_1>})_r+y_{c_r}(X'_{+<y_1>})_r=0\end{aligned}\right\}\qquad(1\text{–}116)$$

$$\left.\begin{aligned}(ky_{c_r}-1)(X_{-<0>})_{r1}+y_{c_r}(X_{-<0>})_{r2}=0\\(ky_{c_r}-1)(X_{-<y_1>})_r+y_{c_r}(X'_{-<y_1>})_r=0\end{aligned}\right\}\qquad(1\text{–}117)$$

(1–116) and (1–117) each corresponds to (1–109′) and (1–110′) respectively. To satisfy (1–116) and (1–117), $y_{c_r}=(1/2)y_1$ is necessary in the reference of each pair of (1–96) and (1–99), and of (1–97) and (1–100). For an example of $y_1=1$, $y_{c_r}=0.5$, a neutral curve is drawn for $k>0$. This curve coincides with Fig. 1 of Miles and Howard (1964), when α used by them is replaced by $k/2$ in the present notations (α is wave number in Miles and Howard, and is estimated with the unit of length $y_1/2$ in the present notations). By this way the instability region ($c_i>0$) exists in the inner area of the neutral curve, if the increase of flow is checked by $U^{(1)}=ay_1$ in I-layer.

For the present treatment we estimate c_i by makig use of (1–102) and (1–103). For the numerical determination of c_i, terms of much higher orders than in estimation of the neutral curve should be taken into consideration.

The characteristic equation from (1–102) and (1–103) is

$$\tan\frac{\delta}{2}\pi(K_1+K_2)(M_1+M_2)+\cot\frac{\delta}{2}\pi(K_1-K_2)(M_1-M_2)=0\qquad(1\text{–}118)$$

To simplify the procedure, $\delta=\frac{1}{2}$ is taken$\left(\text{therefore } R_i=\frac{g\alpha}{a^2}=\frac{3}{16}\right)$, and an example for $y_1=1$, $y_{c_r}=\frac{1}{2}$ and $\alpha=0.02$ is computed numerically to determine both $k(>0)$ and y_{c_i}. With the condition used that $/y_{c_i}/$ is much smaller than y_{c_r} and $y_1-y_{c_r}$, a solution $k\fallingdotseq0.937$, $y_{c_i}\fallingdotseq-0.137$ is obtained. This shows that y_{c_i} is negative, and so c_i is also negative. As it is already found that the neutral curve is not drawn in $\frac{1}{4}>R_i>0$, this means that waves of arbitrary wave number is stable.

As shown by (1–91), y_{c_i}/y_{c_r} and $y_{c_i}/y_1-y_{c_r}$ are not negligible in the present case. The expressions (1–112) and (1–113) of the wave-induced Reynolds stress are also applied to this case, and (1–114) and (1–115) are modified as follows.

$$\tau_{(>y_c)}=\frac{-1}{2}\bar{\rho}a^2A_0^{(2)2}\delta k\frac{e^{2kc_it}}{|(X_{+<0>})_{r1}(X_{-<0>})_{r2}-(X_{-<0>})_{r1}(X_{+<0>})_{r2}|^2}$$
$$\times\left[\sin\delta\pi\left\{\left(ky_{c_r}-\frac{\alpha}{2}y_{c_r}-1\right)(X_{-<0>})_{r1}+y_{c_r}(X_{-<0>})_{r2}\right\}\right.$$
$$\times\left\{\left(ky_{c_r}-\frac{\alpha}{2}y_{c_r}-1\right)(X_{+<0>})_{r1}+y_{c_r}(X_{+<0>})_{r2}\right\}$$

$$+\cos\delta\pi\cdot y_{o_i}\Bigg[\Bigg\{\left(ky_{c_r}-\frac{\alpha}{2}y_{o_r}-1\right)(X_{-<0>})_{r2}+y_{o_r}(X'_{-<0>})_{r2}\Bigg\}$$

$$\times\Bigg\{\left(ky_{c_r}-\frac{\alpha}{2}y_{c_r}-1\right)(X_{+<0>})_{r1}+y_{c_r}(X_{+<0>})_{r2}\Bigg\}$$

$$-\Bigg\{\left(ky_{c_r}-\frac{\alpha}{2}y_{c_r}-1\right)(X_{-<0>})_{r1}+y_{c_r}(X_{-<0>})_{r2}\Bigg\}$$

$$\times\Bigg\{\left(ky_{c_r}-\frac{\alpha}{2}y_{c_r}-1\right)(X_{+<0>})_{r2}+y_{c_r}(X'_{+<0>})_{r2}\Bigg\}$$

$$-(X_{-<0>})_{r2}(X_{+<0>})_{r1}+(X_{-<0>})_{r1}(X_{+<0>})_{r2}\Bigg]\Bigg] \qquad (1\text{-}119)$$

$$\tau_{(<y_c)}=\frac{1}{2}\bar{\rho}a^2A_0^{(2)2}\delta k\frac{e^{2kc_it}}{|(X_{+<0>})_{r1}(X_{-<0>})_{r2}-(X_{-<0>})_{r1}(X_{+<0>})_{r2}|^2}$$

$$\times y_{c_i}\Bigg[\Bigg\{\left(ky_{c_r}-\frac{\alpha}{2}y_{c_r}-1\right)(X_{-<0>})_{r2}+y_{c_r}(X'_{-<0>})_{r2}\Bigg\}$$

$$\times\Bigg\{\left(ky_{c_r}-\frac{\alpha}{2}y_{c_r}-1\right)(X_{+<0>})_{r1}+y_{c_r}(X_{+<0>})_{r2}\Bigg\}$$

$$-\Bigg\{\left(ky_{c_r}-\frac{\alpha}{2}y_{c_r}-1\right)(X_{-<0>})_{r1}+y_{c_r}(X_{-<0>})_{r2}\Bigg\}$$

$$\times\Bigg\{\left(ky_{c_r}-\frac{\alpha}{2}y_{c_r}-1\right)(X_{+<0>})_{r2}+y_{c_r}(X'_{+<0>})_{r2}\Bigg\}$$

$$-(X_{-<0>})_{r2}(X_{+<0>})_{r1}+(X_{-<0>})_{r1}(X_{+<0>})_{r2}\Bigg] \qquad (1\text{-}120)$$

Expressions (1–119) and (1–120) are used for the present numerical example, and the result is as follows.

$$\left.\begin{aligned}\tau_{(>y_c)}&=0.0229\bar{\rho}a^2A_0^{(2)2}e^{2kc_it}\\ \tau_{(<y_c)}&=-0.0477\bar{\rho}a^2A_0^{(2)2}e^{2kc_it}\end{aligned}\right\} \qquad (1\text{-}121)$$

This result clearly shows that the mechanical energy is transferred from the perturbed wave motion to the primary flow, and corresponds to the stable mode of waves.

Another case of stable mode, in which the strong shear flow existed and $1/4>R_i>0$ was kept, was found by K.M. Case (1960). In this example the fixed horizontal boundary was given at $y=0$, and the velocity of general flow was expressed by $U=ay$ ($y>0$), and also the density was expressed by $\bar{\rho}=\rho_0e^{-\alpha y}$. The problem was solved as an initial value problem, and the stable mode was obtained. In the present example the distribution of density is different from Case's example, and the fixed boundary at $y=0$ is replaced by the wavy flexible boundary. The common feature of both examples is a distribution of general flow, which increases monotonously beyond the region where the significant decrease of density appears. It is already shown that, if the velocity of general flow is checked in the upper region, the unstable mode appears in some parts of the wave number-R_i number diagram. By this way in the problem of idealized estuary, which consists of three layers of fresh, salt and mixed water, we can consider the case, in which progressive internal waves are almost stable and do not develop in $1/4>R_i>0$, if the flow increases sufficiently in the fresh water layer. But the influences of the curvature of the velocity profile and of the viscous effect should be reconsidered for further investigations.

2. On the Control Section of Two-layer Flows

2-1. Interfacial linear long wave

We consider a system of two-layer flow, in which $U^{(1)}$ and $U^{(2)}$ mean the uniform velocity of the upper layer flow and of the lower layer flow respectively. Therefore the perturbed wave motion is irrotational, and the linear characteristic equation for wave celerity (inclusive of surface and interfacial waves) can be obtained by dynamical and kinematical conditions of surface, interface and bottom.

$$\begin{aligned}&\{\rho^{(1)}\tanh kh^{(1)}+\rho^{(2)}\coth kh^{(2)}\}c^4-2\{2\rho^{(1)}U^{(1)}\tanh kh^{(1)}+\rho^{(2)}(U^{(1)}\\&+U^{(2)})\coth kh^{(2)}\}c^3+\left[6\rho^{(1)}U^{(1)2}\tanh kh^{(1)}+\rho^{(2)}\{(U^{(1)}+2U^{(2)})^2-3U^{(2)2}\}\right.\\&\left.\times\coth kh^{(2)}-\frac{\rho^{(2)}g}{k}\left\{1+\frac{\tanh kh^{(1)}}{\tanh kh^{(2)}}\right\}\right]c^2-2\left[2\rho^{(1)}U^{(1)3}\tanh kh^{(1)}\right.\\&\left.+\rho^{(2)}U^{(1)}U^{(2)}(U^{(1)}+U^{(2)})\coth kh^{(2)}-\frac{\rho^{(2)}g}{k}\left\{U^{(1)}+U^{(2)}\frac{\tanh kh^{(1)}}{\tanh kh^{(2)}}\right\}\right]c\\&+\left[\rho^{(1)}U^{(1)4}\tanh kh^{(1)}+\rho^{(2)}U^{(1)2}U^{(2)2}\coth kh^{(2)}-\frac{\rho^{(2)}g}{k}\right.\\&\left.\times\left\{U^{(1)2}+U^{(2)2}\frac{\tanh kh^{(1)}}{\tanh kh^{(2)}}\right\}-\left(\frac{g}{k}\right)^2(\rho^{(1)}-\rho^{(2)})\tanh kh^{(1)}\right]=0\end{aligned}\qquad(2\text{-}1)$$

$\rho^{(1)}$ and $\rho^{(2)}$ are the density of upper and lower fluid respectively. $h^{(1)}$ is the thickness of the upper layer, and $h^{(2)}$ is for the lower layer. The bottom is rigid and horizontal, and the surface is free. The algebraic computation for the introduction of (2–1) is cumbersome, but its theoretical foundation is familiar. The solution of (2–1) contains four waves. Two external waves are generally progressive and reverse progressive. Two interfacial waves have much smaller wave celerity than external waves. The external wave is generally stable, but for the interfacial wave the Kelvin-Helmholtz instability may occur in some values of $U^{(1)}$, $U^{(2)}$ and k.

The simplest analysis for the control section of two-layer flow is introduced from (2–1) assuming that $c=0$ and kh_1 and kh_2 are very small. In this case (2–1) is transformed to

$$\begin{aligned}&\rho^{(1)}kh^{(1)}U^{(1)4}+\left(\rho^{(2)}\frac{1}{kh^{(2)}}U^{(2)2}-\frac{\rho^{(2)}g}{k}\right)U^{(1)2}\\&-\frac{\rho^{(2)}g}{k}\frac{h^{(1)}}{h^{(2)}}U^{(2)2}+\left(\frac{g}{k}\right)^2(\rho^{(2)}-\rho^{(1)})kh^{(1)}=0\end{aligned}\qquad(2\text{-}2)$$

(i) Two cases are possible for $U^{(2)}=0$, $U^{(1)}>0$.
The first one is a case of external wave.

$$U^{(1)2}=\frac{\rho^{(2)}g}{\rho^{(1)}k^2h^{(1)}}-\frac{g(\rho^{(2)}-\rho^{(1)})h^{(1)}}{\rho^{(2)}} \tag{2-3}$$

In this case the relation between surface amplitude and the interfacial amplitude is obtained as

$$A^{(i)}=A^{(0)}\left[e^{kh^{(1)}}-\frac{\{g+k(U^{(1)}-c)^2\}}{k(U^{(1)}-c)^2}\sinh kh^{(1)}\right] \tag{2-4}$$

$A^{(0)}$ and $A^{(i)}$ are the amplitude of surface and interface respectively.

By making use of conditions of $c=0$ and $kh^{(1)}$ very small to (2-4), and substituting for $U^{(1)}$ by (2-3),

$$\frac{A^{(i)}}{A^{(0)}}=1-k^2h_1^2\frac{\rho^{(1)}}{\rho^{(2)}}\fallingdotseq 1 \tag{2-5}$$

This means the stationary long wave train externally formed. The distinguished character is from (2-3)

$$\frac{U^{(1)2}}{gh^{(1)}}=\frac{\rho^{(2)}}{\rho^{(1)}}\frac{1}{k^2h^{(1)2}}-\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(2)}} \tag{2-6}$$

This control condition is for the external wave, and is generally far greater than 1.

The second one is for the interfacial wave.

$$\frac{U^{(1)2}}{gh^{(1)}}=\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(2)}},\quad\left(\frac{U^{(1)2}}{g\frac{\Delta\rho}{\rho^{(2)}}h^{(1)}}=F_r^{(1)2}=1\right) \tag{2-7}$$

$$\frac{A^{(0)}}{A^{(i)}}=-\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(1)}} \tag{2-8}$$

The meaning of (2-7) and (2-8) is clear.
(ii) For the case of $U^{(1)}=0$ and $U^{(2)}>0$

$$\frac{U^{(2)2}}{gh^{(2)}}=\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(2)}},\quad\left(\frac{U^{(2)2}}{g\frac{\Delta\rho}{\rho^{(2)}}h^{(2)}}=F_r^{(2)2}=1\right) \tag{2-9}$$

In this case, from the consideration of (2-4), $A^{(0)}=0$ is established for any value of $A^{(i)}$.

The above-mentioned analysis shows that the interfacial long wave as a small amplitude wave can oscillate with a control condition of $F_r^{(1)2}=1$ or $F_r^{(2)2}=1$, and the transition is not one-sided. In some books and papers the formation of the control section of two-layer flow at the river mouth was explained by this mechanism. (G.H. Keulegan (1966) etc.) But the treatment is limited to the small amplitude oscillatory motion. For the long interfacial wave of finite amplitude, the existence of stable steady mode should be examined in each occasion.

2-2. Interfacial hydraulic jump

When the hydraulic jump of idealized type occurs, the depth of flowing layer is not continuous at the control section, and there the jump of finite height may be seen. Two cases of interfacial hydraulic jump are examined with conditions that interfacial and bottom frictions are neglected, and that the surface is free. This may be typical cases of the frictionless aspect of the control section of two-layer flows.

(i) This case consists of a flowing upper layer and a still lower layer. The upper layer is formed by the depth $h^{(1)}$ and the velocity (assuming uniform) $U^{(1)}$ before jump, and $h^{(1)\prime}$ and $U^{(1)\prime}$ after jump. The depth of the lower layer is $h^{(2)}$ and $h^{(2)\prime}$ before and after jump respectivly. The change of the free surface elevation caused by jump is given by Δh (assuming positive when the free surface is lowered by jump). Following to our previous discussions $\Delta\rho=\rho^{(2)}-\rho^{(1)}$ is assumed small as compared with $\rho^{(2)}$.

It is evident that

$$h^{(1)}+h^{(2)}=h^{(1)\prime}+h^{(2)\prime}+\Delta h \tag{2-10}$$

From the law of mass conservation,

$$\rho^{(1)}h^{(1)}U^{(1)}=\rho^{(1)}h^{(1)\prime}U^{(1)\prime} \tag{2-11}$$

Using this, the law of momentum conservation is

$$\frac{g}{2}(h^{(1)}+h^{(1)\prime})\{h^{(1)}-h^{(1)\prime}-(h^{(2)\prime}-h^{(2)})\}=h^{(1)}U^{(1)2}\frac{h^{(1)}-h^{(1)\prime}}{h^{(1)\prime}} \tag{2-12}$$

From the immutability of pressure at the lower layer,

$$(\rho^{(2)}-\rho^{(1)})g(h^{(2)\prime}-h^{(2)})=\rho^{(1)}g\Delta h \tag{2-13}$$

By making use of (2-13), (2-12) is

$$\frac{g}{2}\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(1)}}(h^{(1)}+h^{(1)\prime})(h^{(2)\prime}-h^{(2)})=h^{(1)}U^{(1)2}\frac{h^{(1)}-h^{(1)\prime}}{h^{(1)\prime}} \tag{2-14}$$

From the law of energy conservation,

$$\frac{dE}{dt}+\frac{1}{2}\rho^{(1)}h^{(1)}U^{(1)3}\left(\frac{h^{(1)2}}{h^{(1)\prime 2}}-1\right)-g\rho^{(1)}U^{(1)}h^{(1)}\Delta h=0 \tag{2-15}$$

Here $\frac{dE}{dt}$ should be positive, and it means the energy loss per unit time. Using (2-13), (2-15) is expressed by

$$\frac{dE}{dt}+\frac{1}{2}\rho^{(1)}h^{(1)}U^{(1)3}\left(\frac{h^{(1)2}}{h^{(1)\prime 2}}-1\right)-gU^{(1)}h^{(1)}(\rho^{(2)}-\rho^{(1)})(h^{(2)\prime}-h^{(2)})=0 \tag{2-16}$$

Then (2-14) is considered in more detail. Using (2-10),

$$\frac{g}{2}\frac{\Delta\rho}{\rho^{(1)}}(h^{(1)}+h^{(1)\prime})(h^{(1)}-h^{(1)\prime})-\frac{g}{2}(h^{(1)}+h^{(1)\prime})\frac{\Delta\rho}{\rho^{(1)}}\Delta h$$

$$=h^{(1)}U^{(1)2}\frac{h^{(1)}-h^{(1)\prime}}{h^{(1)\prime}}$$

On the other hand, using the assumption that $\Delta\rho/\rho^{(2)}$ is small, and putting

$(h^{(1)}-h^{(1)\prime})\frac{\Delta\rho}{\rho^{(1)}}\doteqdot\Delta h$ (this is an approximation from (2–13)), the first term of left-hand side of the above relation is

$$\frac{g}{2}(h^{(1)}+h^{(1)\prime})\frac{\Delta\rho}{\rho^{(1)}}(h^{(1)}-h^{(1)\prime})\doteqdot\frac{g}{2}(h^{(1)}+h^{(1)\prime})\frac{\Delta\rho}{\rho^{(2)}}(h^{(1)}-h^{(1)\prime})$$

$$+\frac{g}{2}(h^{(1)}+h^{(1)\prime})\frac{\Delta\rho}{\rho^{(1)}}\Delta h$$

By this way (2–14) is transformed to

$$\frac{g}{2}\frac{\Delta\rho}{\rho^{(2)}}(h^{(1)}+h^{(1)\prime})=\frac{h^{(1)}}{h^{(1)\prime}}U^{(1)2} \tag{2-17}$$

From (2–17) we can easily obtain

$$\frac{h^{(1)\prime}}{h^{(1)}}=-\frac{1}{2}+\frac{1}{2}\sqrt{1+8F_r^{(1)2}}\ , \quad F_r^{(1)2}=\frac{U^{(1)2}}{\frac{\Delta\rho}{\rho^{(2)}}gh^{(1)}} \tag{2-18}$$

This is firstly obtained by C.S. Yih and C.R. Guha (1955). This relation shows with sufficient approximation that $h^{(1)\prime}/h^{(1)}\gtrless1$ consists in correspondence with $F_r^{(1)2}\gtrless1$. Therefore, from the low of momentum conservation, in which frictional effect is neglected, both cases of $h^{(1)\prime}/h^{(1)}>1$ and of $h^{(1)\prime}/h^{(1)}<1$ seem to be possible according to the value of $F_r^{(1)2}$. At this point the relation (2–16) is to be examined. By making use of (2–14), (this is a strict form of momentum relation),

$$\frac{dE}{dt}=-gU^{(1)}\Delta\rho(h^{(2)\prime}-h^{(2)})\frac{(h^{(1)}-h^{(1)\prime})^2}{4h^{(1)\prime}}$$

$$=-gU^{(1)}\Delta\rho(h^{(1)}-h^{(1)\prime}-\Delta h)\frac{(h^{(1)}-h^{(1)\prime})^2}{4h^{(1)\prime}}\geq0 \tag{2-19}$$

(2–19) is a correct energy condition, in which the frictional loss is neglected, and from this $h^{(2)\prime}-h^{(2)}\leq0$ is realized. Also in the reference of (2–13), Δh becomes negative, indicating the rise up of the surface at the downstream of the jump. Accordingly $h^{(1)}-h^{(1)\prime}<0$ is evident. Thus the consideration of energy conservation limits the possible case to $h^{(1)\prime}/h^{(1)}\geq1$ (Fig. 5), and this means that the flow should be from supercritical to subcritical. The transition at the control section is one-sided, indicating that the loss of energy (2–19) at the jump is not negligible in the treatment of finite height.

(ii) A case, in which the upper layer is still, and the lower layer flows, is taken into consideration. The thickness of each layer before and after jump is expressed with the same notation as (i). The velocity of the lower layer before jump is noted as $U^{(2)}$, and $U^{(2)\prime}$ for the velocity after jump. As the statical pressure in the upper layer is not varied (there is not the frictional effect of interface and bottom), the relation, which corresponds to (2–10) of (i), is

$$h^{(1)}-h^{(1)\prime}=h^{(2)\prime}-h^{(2)} \tag{2-20}$$

The relation of momentum conservation, which corresponds to (2–18) of (i), is

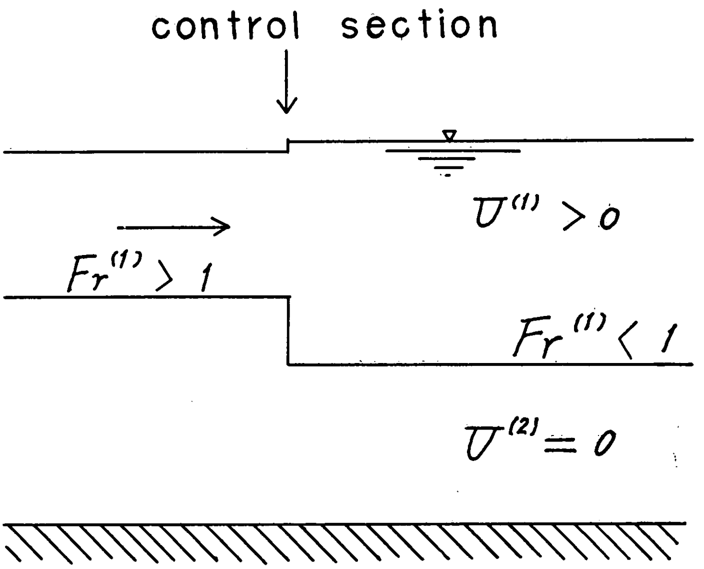

Fig. 5. Interfacial hydraulic jump

$$\frac{h^{(2)\prime}}{h^{(2)}}=-\frac{1}{2}+\frac{1}{2}\sqrt{1+8F_r^{(2)2}}\ ,\quad F_r^{(2)2}=\frac{U^{(2)2}}{\frac{\Delta\rho}{\rho^{(2)}}gh^{(2)}} \tag{2-21}$$

The relation of energy conservation, which corresponds to (2-19) of (i), is

$$\frac{dE}{dt}=\Delta\rho g U^{(2)}\frac{(h^{(2)\prime}-h^{(2)})^3}{4h^{(2)\prime}}\geq 0 \tag{2-22}$$

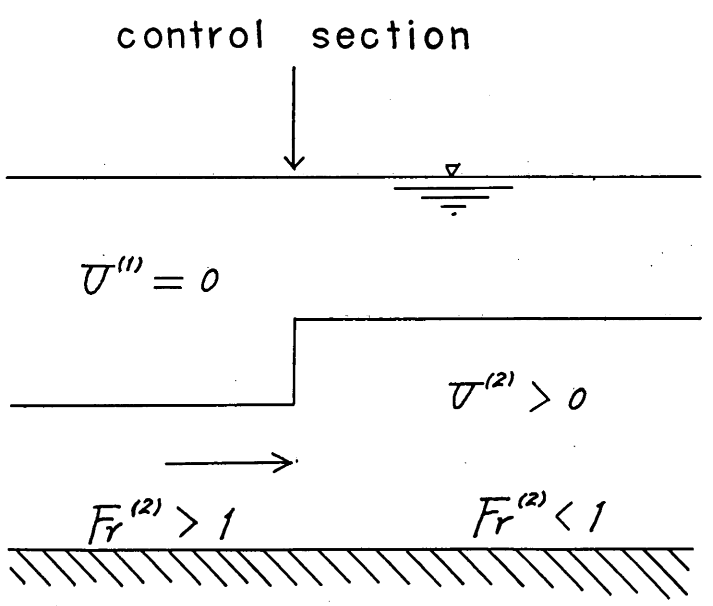

Fig. 6. Interfacial hydraulic jump

From (2-22) $h^{(2)\prime}>h^{(2)}$ is immediately obtained, and (2-21) indicates that $F_r^{(2)2}$ is greater than 1 before jump, and it becomes smaller than 1 after jump. The tendency from supercritical to subcritical is same as (i), and the transition is again one-sided (Fig. 6).

2-3. The control section with variable width and interfacial resistance

In this section the interfacial resistance and the bottom resistance (if necessary) are taken into account, and stillmore the variation of the width of the flow is also considered.

(i) A case of $U^{(2)}=0$ and $U^{(1)}>0$ is treated. A suitable equation of motion was already given by J.B. Schijf and J.C. Schönfeld (1953).

For the upper layer

$$\left.\begin{aligned}\rho^{(1)}U^{(1)}\frac{\partial U^{(1)}}{\partial x}&=-\frac{\partial p}{\partial x}+\frac{\partial \tau}{\partial y}\\ 0&=-\rho^{(1)}g-\frac{\partial p}{\partial y}\end{aligned}\right\}\qquad(2\text{-}23)$$

For the lower layer

$$\left.\begin{aligned}0&=-\frac{\partial p}{\partial x}+\frac{\partial \tau}{\partial y}\\ 0&=-\rho^{(2)}g-\frac{\partial p}{\partial y}\end{aligned}\right\}\qquad(2\text{-}24)$$

The equation of continuity is

$$Q^{(1)}=U^{(1)}h^{(1)}b^{(1)}=\text{const}\qquad(2\text{-}25)$$

Here x is abscissa taken horizontally and is downstream positive. y is ordinate upward positive. $b^{(1)}$ is the width of flow (assuming $b^{(1)}=b^{(2)}$ implicitly), and the bottom is taken horizontally. From (2-23) and (2-24) following relations are obtained using the shearing stress of interface τ_i.

For the flowing upper layer,

$$\frac{1}{2}\frac{\partial U^{(1)2}}{\partial x}=-g\frac{\partial h^{(1)}}{\partial x}-g\frac{\partial h^{(2)}}{\partial x}-\frac{\tau_i}{\rho^{(1)}h^{(1)}}\qquad(2\text{-}26)$$

For the still lower layer,

$$-\frac{\rho^{(1)}}{\rho^{(2)}}g\frac{\partial h^{(1)}}{\partial x}-g\frac{\partial h^{(2)}}{\partial x}+\frac{\tau_i}{\rho^{(2)}h^{(2)}}=0\qquad(2\text{-}27)$$

In this case stress acting on the bottom base is neglected, and in actual problem it contains some questions.

Eliminating $g\dfrac{\partial h^{(2)}}{\partial x}$ from (2-26) and (2-27), and using the relation

$$\frac{\partial U^{(1)2}}{\partial x}=-2\frac{Q^{(1)2}}{h^{(1)2}b^{(1)2}}\left(\frac{\frac{\partial h^{(1)}}{\partial x}}{h^{(1)}}+\frac{\frac{\partial b^{(1)}}{\partial x}}{b^{(1)}}\right)$$

we obtain

$$\left(1-\frac{Q^{(1)2}}{h^{(1)2}b^{(1)2}}\frac{1}{\varepsilon g h^{(1)}}\right)\frac{\partial h^{(1)}}{\partial x}-\frac{Q^{(1)2}}{h^{(1)2}b^{(1)2}}\frac{\frac{\partial b^{(1)}}{\partial x}}{\varepsilon g b^{(1)}}+\frac{\tau_i}{\rho}\frac{1}{\varepsilon g}\frac{h^{(1)}+h^{(2)}}{h^{(1)}h^{(2)}}=0 \qquad (2\text{–}28)$$

Here $\varepsilon=\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(2)}}$, and a notation ρ of the third term is used as a common representation of $\rho^{(1)}$ and $\rho^{(2)}$, as the difference between $\rho^{(2)}$ and $\rho^{(1)}$ is small.

The interfacial shear stress τ_i has a complicated character, as it has not the usual property of shear stress acting on the rigid boundary. The amplification of interfacial wave may be a cause of this stress, and in this case the exchange of momentum depends mainly upon the action of pressure. Therefore the detailed character of τ_i may be examined in the future problem. In this paper we put it as $\tau_i=\rho\kappa U^{(1)2}$ in the similar manner of the general expression of flow resistance. This type of expression is also used to the shear stress of air flow upon water surface covered by developing wind waves.

Using this expression (2–28) is

$$\left(1-\frac{1}{\varepsilon g}\frac{Q^{(1)2}}{b^{(1)2}h^{(1)3}}\right)\frac{\partial h^{(1)}}{\partial x}-\frac{1}{\varepsilon g}\frac{Q^{(1)2}}{h^{(1)2}b^{(1)3}}\frac{\partial b^{(1)}}{\partial x}+\frac{\kappa}{\varepsilon g}\frac{h^{(1)}+h^{(2)}}{h^{(2)}}\frac{Q^{(1)2}}{b^{(1)2}h^{(1)3}}=0 \qquad (2\text{–}29)$$

The relation (2–29) is perturbed in the vicinity of $\frac{U^{(1)2}}{\varepsilon g h^{(1)}}=F_r^{(1)2}=1$. In this perturbation the variation of $h^{(1)}+h^{(2)}=h$ is very small, and it is neglected. As the density of fresh water is very near to the density of salt water in estuarial problems, this simplification does not influence the fundamental characteristic of the perturbation.

The first relation is shown by

$$-\frac{h_0^{(1)}}{b_0^{(1)}}\left(\frac{\partial b^{(1)}}{\partial x}\right)_0+\frac{\kappa(h^{(1)}+h^{(2)})}{h_0^{(2)}}=0 \qquad (2\text{–}30)$$

The second relation is shown by

$$\frac{3}{h_0^{(1)}}\left(\frac{\partial h^{(1)}}{\partial x}\right)_0^2+\left\{\frac{2}{b_0^{(1)}}\left(\frac{\partial b^{(1)}}{\partial x}\right)_0+\frac{\kappa h}{h_0^{(1)}}\frac{2h_0^{(1)}-h}{h_0^{(2)2}}\right\}$$
$$\times\left(\frac{\partial h^{(1)}}{\partial x}\right)_0+\frac{\kappa h}{h_0^{(2)}}\left\{\frac{\left(\frac{\partial b^{(1)}}{\partial x}\right)_0}{b_0^{(1)}}-\frac{\left(\frac{\partial^2 b^{(1)}}{\partial x^2}\right)_0}{\left(\frac{\partial b^{(1)}}{\partial x}\right)_0}\right\}=0 \qquad (2\text{–}31)$$

h is equal to $h^{(1)}+h^{(2)}$ as referred, and a suffix 0 means the value at the control section. From (2–30) we can put $b^{(1)}=b_0^{(1)}e^{\alpha x}$ in the vicinity of the control section, and so

$$\alpha=\frac{\kappa h}{h_0^{(1)}h_0^{(2)}} \qquad \left(\alpha=\frac{1}{b_0^{(1)}}\left(\frac{\partial b^{(1)}}{\partial x}\right)_0\right)>0 \qquad (2\text{–}32)$$

From (2–31)

$$\frac{1}{h_0^{(1)}}\left(\frac{\partial h^{(1)}}{\partial x}\right)_0=-\frac{1}{3}\frac{1}{b_0^{(1)}}\left(\frac{\partial b^{(1)}}{\partial x}\right)_0\frac{h}{h_0^{(2)}}=-\frac{1}{3}\frac{\kappa h^2}{h_0^{(1)}h_0^{(2)2}}<0 \tag{2-33}$$

(2-32) indicates the increase of width at the control section, and (2-33) gives the decrease of the thickness of the flowing upper layer. The transition at the control section is different from the case of hydraulic jump of 2-2.

Generally the surface slope is very small compared with the slope of the interface given by (2-33), and its approximate value is determined from (2-26) and (2-27) by the elimination of τ_i. Using $F_r^{(1)2}=\dfrac{U^{(1)2}}{\varepsilon g h^{(1)}}$ and $F_r^{(\text{ext.})2}=\dfrac{U^{(1)2}}{g h^{(1)}}$ the gradient of water surface $\left(\dfrac{\partial h}{\partial x}\text{ in the present model}\right)$ is expressed in general by

$$\frac{\partial h}{\partial x}=\frac{\partial h^{(2)}}{\partial x}\frac{-\varepsilon\left(\dfrac{1}{\rho^{(1)}h^{(1)}}+\dfrac{F_r^{(1)2}}{\rho^{(2)}h^{(2)}}\right)}{\dfrac{1-\varepsilon}{\rho^{(1)}h^{(1)}}+\dfrac{1-F_r^{(\text{ext.})2}}{\rho^{(2)}h^{(2)}}}+\frac{1}{b^{(1)}}\frac{\partial b^{(1)}}{\partial x}\frac{\dfrac{U^{(1)2}}{\rho^{(2)}gh^{(2)}}}{\dfrac{1-\varepsilon}{\rho^{(1)}h^{(1)}}+\dfrac{1-F_r^{(\text{ext.})2}}{\rho^{(2)}h^{(2)}}} \tag{2-34}$$

At the control section $F_r^{(1)2}=1$ and $F_r^{(\text{ext.})2}=\varepsilon$ are clear, and by making use of (2-32),

$$\left(\frac{\partial h}{\partial x}\right)_0=\left(\frac{\partial h^{(2)}}{\partial x}\right)_0\frac{-\varepsilon}{1-\varepsilon}+\frac{\varepsilon}{1-\varepsilon}\frac{h_0^{(1)}}{h_0^{(2)}}\kappa \tag{2-35}$$

Using (2-33), (2-35) is approximated by

$$\left(\frac{\partial h}{\partial x}\right)_0=-\frac{1}{3}\kappa\frac{\varepsilon}{1-\varepsilon}\left\{\left(\frac{h_0^{(1)}}{h_0^{(2)}}-1\right)^2+\frac{h_0^{(1)}}{h_0^{(2)}}\right\}<0 \tag{2-36}$$

$\left(\dfrac{\partial h}{\partial x}\right)_0$ is very small compared with $\left(\dfrac{\partial h^{(1)}}{\partial x}\right)_0$ of (2-33), because $\dfrac{\varepsilon}{1-\varepsilon}\ll 1$.

A noticeable point is that the coefficient of interfacial friction κ is included linearly in (2-32), (2-33) and (2-36), thus regulating the increase of width, the decrease of the depth of the upper layer and the surface gradient at the control section. Though the true property of κ seems to have many difficult points, its experimental determination was already done by both model and field observations of the two-layer flow of fresh and salt water. The value of κ is about the order of 10^{-2} by model tests, and about $10^{-3}\sim 10^{-4}$ by field observations. Accordingly we can apply to a certain extent the present theory to the steady two-layer flow at the rivermouth, where the flow of fresh water runs out to sea on a saline wedge (Fig. 7).

(ii) Next we treat a case of $U^{(1)}=0$ and $U^{(2)}>0$.

Equations of motion for the upper layer are

$$\left.\begin{aligned}0&=-\frac{\partial p}{\partial x}+\frac{\partial \tau}{\partial y}\\0&=-\rho^{(1)}g-\frac{\partial p}{\partial y}\end{aligned}\right\} \tag{2-37}$$

and for the lower layer

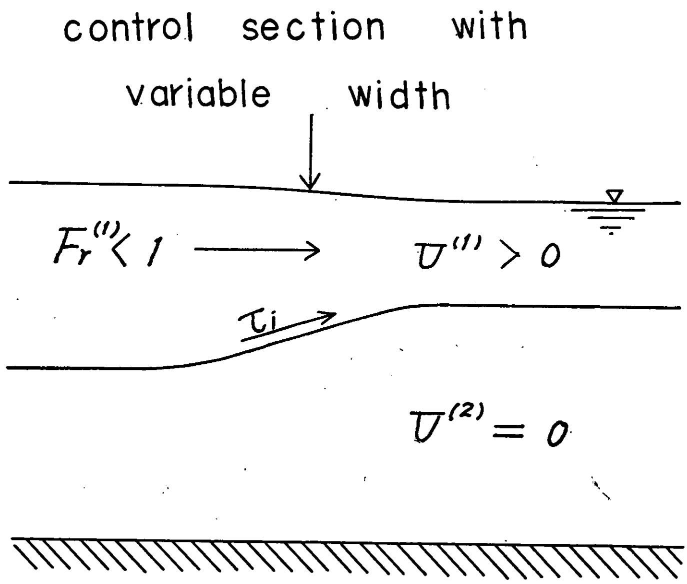

Fig. 7. Control section with variable width and interfacial resistance

$$\left.\begin{aligned} \rho^{(2)}U^{(2)}\frac{\partial U^{(2)}}{\partial x} &= -\frac{\partial p}{\partial x}+\frac{\partial \tau}{\partial y} \\ 0 &= -\rho^{(2)}g-\frac{\partial p}{\partial y} \end{aligned}\right\} \qquad (2\text{–}38)$$

The equation of continuity is

$$Q^{(2)}=U^{(2)}a^{(2)}b^{(2)}=\text{const.} \qquad (2\text{–}39)$$

In the present case we must take into account the interfacial resistance τ_i and the bottom resistance τ_b. Because $\tau_i<0$ and $\tau_b>0$, we put them as follows in a similar form of (i).

$$\tau_i=-\rho^{(2)}\kappa_i U^{(2)2}, \quad \tau_b=\rho^{(2)}\kappa_b U^{(2)2} \quad (\kappa_i \text{ and } \kappa_b>0) \qquad (2\text{–}40)$$

After some computations the next relation is obtained, and it corresponds to (2–29) of (i).

$$\left(1-\frac{1}{\varepsilon g h^{(2)}}\frac{Q^{(2)2}}{h^{(2)2}b^{(2)2}}\right)\frac{\partial h^{(2)}}{\partial x}-\frac{\frac{\partial b^{(2)}}{\partial x}}{\varepsilon g b^{(2)}}\frac{Q^{(2)2}}{h^{(2)2}b^{(2)2}}$$
$$+\frac{\kappa_i}{\varepsilon g h^{(2)}}\frac{Q^{(2)2}}{h^{(2)2}b^{(2)2}}\frac{h}{h^{(1)}}+\frac{\kappa_b}{\varepsilon g h^{(2)}}\frac{Q^{(2)2}}{h^{(2)2}b^{(2)2}}=0 \qquad (2\text{–}41)$$

(2–41) is perturbed in the vicinity of the control section $\frac{U^{(2)2}}{\varepsilon g h^{(2)}}=F_r^{(2)2}=1$. The change of total depth $h=h^{(1)}+h^{(2)}$ is neglected. The first and the second relations are

$$-\frac{h_0^{(2)}}{b_0^{(2)}}\left(\frac{\partial b^{(2)}}{\partial x}\right)_0+\kappa_i\frac{h}{h_0^{(1)}}+\kappa_b=0 \tag{2-42}$$

$$\frac{3}{h_0^{(2)}}\left(\frac{\partial h^{(2)}}{\partial x}\right)_0^2+\left\{\frac{1}{b_0^{(2)}}\left(\frac{\partial b^{(2)}}{\partial x}\right)_0+\kappa_i\frac{h}{h_0^{(1)2}}\right\}$$
$$\times\left(\frac{\partial h^{(2)}}{\partial x}\right)_0-\frac{h_0^{(2)}}{b_0^{(2)}}\left(\frac{\partial^2 b^{(2)}}{\partial x^2}\right)_0+\frac{h_0^{(2)}}{b_0^{(2)2}}\left(\frac{\partial b^{(2)}}{\partial x}\right)_0^2=0 \tag{2-43}$$

Putting $b^{(2)}=b_0^{(2)}e^{\beta x}$, and from (2-42) and (2-43)

$$\beta=\kappa_i\frac{h}{h_0^{(2)}h_0^{(1)}}+\kappa_b\frac{1}{h_0^{(2)}} \tag{2-44}$$

$$\left(\frac{\partial h^{(2)}}{\partial x}\right)_0=-\frac{1}{3}\left(\kappa_i\frac{h^2}{h_0^{(1)2}}+\kappa_b\right) \tag{2-45}$$

The surface slope (very small quantity than the interfacial slope) is obtained by the elimination of τ_i from equations of motion and continuity. In the general case it is

$$\frac{\partial h}{\partial x}=\frac{\varepsilon}{1-\varepsilon}\frac{h^{(2)}}{h}\left\{\left(\frac{U^{(2)2}}{\varepsilon g h^{(2)}}-1\right)\frac{\partial h^{(2)}}{\partial x}-\frac{\tau_b}{\rho^{(2)}\varepsilon g h^{(2)}}+\frac{h^{(2)}U^{(2)2}}{\varepsilon g h^{(2)}}\frac{\frac{\partial b^{(2)}}{\partial x}}{b^{(2)}}\right\} \tag{2-46}$$

At the condition of $\dfrac{U^{(2)2}}{\varepsilon g h^{(2)}}=F_r^{(2)2}=1$,

$$\left(\frac{\partial h}{\partial x}\right)_0=\kappa_i\frac{h_0^{(2)}}{h_0^{(1)}}\frac{\varepsilon}{1-\varepsilon} \tag{2-47}$$

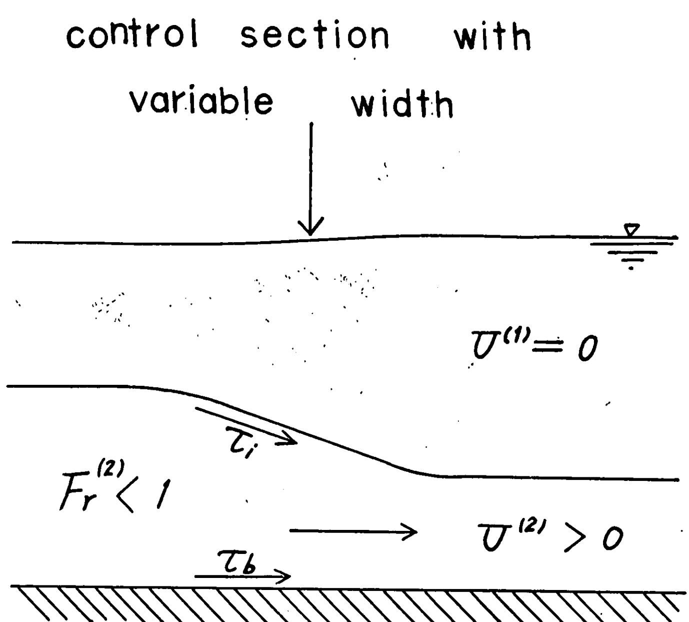

Fig. 8. Control section with variable width and interfacial and bottom resistance

(2–44) shows the increase of the width at the control section, and (2–45) shows the decrease of the depth of flowing lower layer. Stillmore (2–47) indicates the ascent of surface elevation ot the same section. (it is interesting that this ascent is controlled by the shearing stress at the interface only). These properties are different from the interfacial jump of (ii) of 2–2, and κ_i and κ_b play a role of linear regulation. This case of an undercurrent may be seen in practice by the flowing muddy water under clean still water, if the width of flow is suddenly enlarged (Fig. 8).

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