REPORT

OF

PORT AND HARBOUR RESEARCH INSTITUTE

REPORT NO. 14

The Problems of Density Current
Part, I

by

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December, 1967

PORT AND HARBOUR RESEARCH INSTITUTE
MINISTRY OF TRANSPORT

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THE PROBLEMS OF DENSITY CURRENT PART, I

Contents

密 度 流 の 問 題

浜　田　徳　一

要　旨：

2 層よりなる密度流の基本的問題点である界面内波の諸性質を調べたものであり，粘性損失，境界層の安定，有限振幅波における諸性質，砕波の問題および古典的な内波増幅機構としての Kelvin-Helmholtz 不安定の諸特性について検討している。

THE PROBLEMS OF DENSITY CURRENT
PART, I

Tokuichi Hamada

Synopsis:

In this paper three problems of interfacial waves at the boundary of two homogeneous liquids are discussed. As the aim of this paper concentrates mainly to clarify the dynamical characteristics of interfacial problems of salt and fresh water, the properties of two liquids are confined to these in some cases of treatment.

In chapter 1 the viscous boundary layer of the interfacial wave between salt and fresh water is discussed. A very simple expression of the coefficient of viscous attenuation of waves is obtained, and the characteristics of the structure of interfacial boundary layer is discussed. The result shows the possibility of the breakdown of the laminar layer caused by the viscous instability.

In chapter 2 the properties of interfacial waves of finite amplitude is examined. The perturbation method by the slope of wave profile is used, and an ambiguity in the approximation of the third order is pointed out. Then the possibility of the inviscid breaking of the interfacial wave of the permanent type is discussed. The result shows the impossibility of breaking, when the density of upper and lower layers is comparable.

In chapter 3 the detailed discussion of the Kelvin-Helmholtz instability is expressed. At the first order approximation the dynamics of energy transfer between the no-perturbed flow and the perturbed wave is examined. It is different from that used by J. W. Miles (1959) in general. At the second order approximation the properties of wave profile are discussed. The characteristics of the instability is obtained at the limit when the density of both layers approaches each other boundlessly.

1. Viscous dissipation of interfacial waves

1.1 Characteristic equation

In this chapter we pursue some dynamical properties of viscous dissipation of interfacial waves observed at the horizontal boundary of salt and fresh water in stationary state. The main concept of this analysis may be extended to the instability problems of interfacial boundary of usual estuaries. Some computation (G. H. Keulegan, 1949) was already given for this problem, but the treatment was an application of simpler problem of surface waves and was not conclusive. The present computation is a treatment approximated pertinently, in which important factors for this boundary layer problem are carefully taken into account.

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We consider the case of two-dimensional space at the incompressible state, and the first order approximation of wave equations is treated. x-axis (abscissa) is taken horizontally along the interfacial boundary of both fluids in still condition, and y-axis (ordinate) is taken vertically upwards positive. $\varphi^{(1)}, \phi^{(1)}, u^{(1)}, \cdots$ are notations concerned with the upper fluid, and $\varphi^{(2)}, \phi^{(2)}, u^{(2)}, \cdots$ corresponds to the lower fluid.

In a reference of H. Lamb (1932), equations of motion and continuity of both fluids are

$$\frac{\partial u^{(1)}}{\partial t}=-\frac{1}{\rho^{(1)}}\frac{\partial p^{(1)}}{\partial x}+\nu^{(1)}\nabla^2 u^{(1)} \tag{1-1-1}$$

$$\frac{\partial v^{(1)}}{\partial t}=-\frac{1}{\rho^{(1)}}\frac{\partial p^{(1)}}{\partial y}-g+\nu^{(1)}\nabla^2 v^{(1)} \tag{1-1-2}$$

$$\frac{\partial u^{(1)}}{\partial x}+\frac{\partial v^{(1)}}{\partial y}=0 \tag{1-1-3}$$

$$u^{(1)}=-\frac{\partial \varphi^{(1)}}{\partial x}-\frac{\partial \phi^{(1)}}{\partial y}, \qquad v^{(1)}=-\frac{\partial \varphi^{(1)}}{\partial y}+\frac{\partial \phi^{(1)}}{\partial x}. \tag{1-1-4}$$

From the above relations, $\varphi^{(1)}$ and $\phi^{(1)}$ are

$$\frac{p^{(1)}}{\rho^{(1)}}=\frac{\partial \varphi^{(1)}}{\partial t}-gy \tag{1-1-5}$$

$$\nabla^2\varphi^{(1)}=0, \qquad \frac{\partial \phi^{(1)}}{\partial t}=\nu^{(1)}\nabla^2\phi^{(1)} \tag{1-1-6}$$

....

$$\frac{\partial u^{(2)}}{\partial t}=-\frac{1}{\rho^{(2)}}\frac{\partial p^{(2)}}{\partial x}+\nu^{(2)}\nabla^2 u^{(2)} \tag{1-2-1}$$

$$\frac{\partial v^{(2)}}{\partial t}=-\frac{1}{\rho^{(2)}}\frac{\partial p^{(2)}}{\partial y}-g+\nu^{(2)}\nabla^2 v^{(2)} \tag{1-2-2}$$

$$\frac{\partial u^{(2)}}{\partial x}+\frac{\partial v^{(2)}}{\partial y}=0 \tag{1-2-3}$$

$$u^{(2)}=-\frac{\partial \varphi^{(2)}}{\partial x}-\frac{\partial \phi^{(2)}}{\partial y}, \qquad v^{(2)}=-\frac{\partial \varphi^{(2)}}{\partial y}+\frac{\partial \phi^{(2)}}{\partial x}. \tag{1-2-4}$$

From these relations,

$$\frac{p^{(2)}}{\rho^{(2)}}=\frac{\partial \varphi^{(2)}}{\partial t}-gy \tag{1-2-5}$$

$$\nabla^2\varphi^{(2)}=0, \qquad \frac{\partial \phi^{(2)}}{\partial t}=\nu^{(2)}\nabla^2\phi^{(2)}. \tag{1-2-6}$$

The dynamical boundary condition at the interface are as follows under the assumption that waves are small amplitude.

$$(p_{yy})^{(2)}=(p_{yy})^{(1)}+T\frac{\partial^2\eta}{\partial x^2}$$

or

$$-p^{(2)}+2\mu^{(2)}\frac{\partial v^{(2)}}{\partial y}=T\frac{\partial^2\eta}{\partial x^2}-p^{(1)}+2\mu^{(1)}\frac{\partial v^{(1)}}{\partial y} \quad \text{at} \quad y=0 \tag{1-3}$$

$$(\tau_{xy})^{(1)}=(\tau_{xy})^{(2)}$$

or

$$\mu^{(1)}\left(\frac{\partial v^{(1)}}{\partial x}+\frac{\partial u^{(1)}}{\partial y}\right)=\mu^{(2)}\left(\frac{\partial v^{(2)}}{\partial x}+\frac{\partial u^{(2)}}{\partial y}\right) \quad \text{at} \quad y=0 \tag{1-4}$$

Two kinematic conditions at the interface are

$$v^{(1)}=v^{(2)}=\frac{\partial\eta}{\partial t} \quad \text{at} \quad y=0 \tag{1-5-1}$$

$$u^{(1)}=u^{(2)} \quad \text{at} \quad y=0 \tag{1-5-2}$$

In these equations $\varphi^{(i)}$ and $\phi^{(i)}$ $(i=1,2)$ indicate the inviscid (slowly varying) solutions and viscid (rapidly varying) solutions respectively, and they can be combinated linearly in the present approximation of the first order.

We assume a sinusoidal progressive oscillation.

$$\varphi^{(1)}=A^{(1)}e^{i(kx+nt)}, \qquad \phi^{(1)}=B^{(1)}e^{i(kx+nt)}$$

and so $k>0$, $\boldsymbol{R}(n)<0$ is clear.

Using $m^{(1)}$, which has the positive real part and satisfies the next relation,

$$m^{(1)2}=k^2+\frac{in}{\nu^{(1)}} \tag{1-6}$$

(1-1-6) may be satisfied by

$$\left.\begin{aligned}\varphi^{(1)}&=A_2^{(1)}e^{-ky}\cdot e^{i(kx+nt)}\\ \phi^{(1)}&=B_2^{(1)}e^{-m^{(1)}y}\cdot e^{i(kx+nt)}\end{aligned}\right\} \tag{1-7}$$

Similarly using $m^{(2)}$, which has the same property with $m^{(1)}$,

$$m^{(2)2}=k^2+\frac{in}{\nu^{(2)}} \tag{1-8}$$

$$\left.\begin{aligned}\varphi^{(2)}&=A_1^{(2)}\,e^{ky}\,e^{i(kx+nt)}\\ \phi^{(2)}&=B_1^{(2)}\,e^{m^{(2)}y}\,e^{i(kx+nt)}\end{aligned}\right\} \tag{1-9}$$

We assume the vertical displacement of the interface as

$$\eta=De^{i(kx+nt)} \tag{1-10}$$

(hare D is taken as real and positive)

Expressions of $p^{(1)}$ and $p^{(2)}$ of (1-3) are found in (1-1-5) and (1-2-5), and, by making use of expressions given by (1-7), (1-9) and (1-10), from (1-3);

$$\begin{aligned}&\rho^{(2)}gD-\rho^{(2)}inA_1^{(2)}-2\mu^{(2)}k^2A_1^{(2)}+2\mu^{(2)}ikm^{(2)}B_1^{(2)}\\ &=-k^2TD+\rho^{(1)}gD-\rho^{(1)}inA_2^{(1)}-2\mu^{(1)}k^2A_2^{(1)}-2\mu^{(1)}im^{(1)}kB_2^{(1)}\end{aligned} \tag{1-11}$$

from (1−4),

$$\mu^{(1)}(ik^2A_2^{(1)}-k^2B_2^{(1)}+ik^2A_2^{(1)}-m^{(1)2}B_2^{(1)})$$
$$=\mu^{(2)}(-ik^2A_1^{(2)}-k^2B_1^{(2)}-ik^2A_1^{(2)}-m^{(2)2}B_1^{(2)}) \qquad (1\text{-}12)$$

from (1–5–1)

$$kA_2^{(1)}+ikB_2^{(1)}=-kA_1^{(2)}+ikB_1^{(2)}=inD \qquad (1\text{-}13)$$

from (1–5–2)

$$-ikA_2^{(1)}+m^{(1)}B_2^{(1)}=-ikA_1^{(2)}-m^{(2)}B_1^{(2)} \qquad (1\text{-}14)$$

Using (1–11), (1–12), (1–13) and (1–14), we can obtain the eigenvalue equation of n as follows.

$$(\rho^{(2)}g+k^2T-\rho^{(1)}g)\frac{1}{2n}[\{\mu^{(1)}(k^2+m^{(1)2})+\mu^{(2)}k(m^{(1)}-k)$$
$$-\mu^{(1)}k(m^{(1)}+k)\}(k-m^{(2)})+\{\mu^{(2)}(k^2+m^{(2)2})-\mu^{(1)}k(k-m^{(2)})$$
$$-\mu^{(2)}k(m^{(2)}+k)\}(k-m^{(1)})]-(\rho^{(2)}in+2\mu^{(2)}k^2)\frac{i}{2k}[\{\mu^{(1)}(k^2+m^{(1)2})$$
$$+\mu^{(2)}k(m^{(1)}-k)-\mu^{(1)}k(m^{(1)}+k)\}(m^{(2)}+k)+\{\mu^{(2)}(k^2+m^{(2)2})$$
$$-\mu^{(1)}k(k-m^{(2)})-\mu^{(2)}k(m^{(2)}+k)\}(m^{(1)}-k)]$$
$$+2\mu^{(2)}ikm^{(2)}\{\mu^{(1)}(k^2+m^{(1)2})+\mu^{(2)}k(m^{(1)}-k)-\mu^{(1)}k(m^{(1)}+k)\}$$
$$+(\rho^{(1)}in+2\mu^{(1)}k^2)\frac{i}{2k}[\{\mu^{(1)}(k^2+m^{(1)2})+\mu^{(2)}k(m^{(1)}-k)$$
$$-\mu^{(1)}k(m^{(1)}+k)\}(k-m^{(2)})-\{\mu^{(2)}(k^2+m^{(2)2})-\mu^{(1)}k(k-m^{(2)})$$
$$-\mu^{(2)}k(m^{(2)}+k)\}(m^{(1)}+k)]+2\mu^{(1)}im^{(1)}k\{\mu^{(2)}(k^2+m^{(2)2})$$
$$-\mu^{(1)}k(k-m^{(2)})-\mu^{(2)}k(m^{(2)}+k)\}=0 \qquad (1\text{-}15)$$

The general treatment of (1–15) seems difficult (S. Chandrasekhar (1961), C.—M. Tchen (1956) etc.), but in the present problem we can estimate the value of n in the next section by making use of numerical data of both fluids.

1.2 The case in which the viscosity of both fluids is nearly equal

Some consideration shows that, in the present problem of two layers of salt and fresh water, the relation of (1–15) may be approximately solved as the case of same coefficient of molecular viscosity, so long as the temperature of both fluids is moderate and the difference is not so large.

Accordingly the following approximations may be held in (1–15),

$$\mu^{(1)}k^2+\mu^{(1)}k^2+\rho^{(1)}in+\mu^{(2)}km^{(1)}-\mu^{(2)}k^2-\mu^{(1)}km^{(1)}-\mu^{(1)}k^2\fallingdotseq\rho^{(1)}in$$
$$\mu^{(2)}k^2+\mu^{(2)}k^2+\rho^{(2)}in-\mu^{(1)}k^2+\mu^{(1)}km^{(2)}-\mu^{(2)}km^{(2)}-\mu^{(2)}k^2\fallingdotseq\rho^{(2)}in$$

and so (1–15) may be simplified to

$$(\rho^{(2)}g+k^2T-\rho^{(1)}g)-\frac{n^2}{k}(\rho^{(2)}+\rho^{(1)})+2\mu^{(2)}kin+2\mu^{(1)}kin$$
$$-\frac{4\mu^{(2)}km^{(2)}\rho^{(1)}in}{\rho^{(1)}m^{(2)}+\rho^{(2)}m^{(1)}}-\frac{4\mu^{(1)}m^{(1)}k\rho^{(2)}in}{\rho^{(1)}m^{(2)}+\rho^{(2)}m^{(1)}}$$

$$-\frac{(\rho^{(2)}g+k^2T-\rho^{(1)}g)(\rho^{(1)}+\rho^{(2)})k}{\rho^{(1)}m^{(2)}+\rho^{(2)}m^{(1)}}+\frac{n^2(\rho^{(1)}-\rho^{(2)})^2}{\rho^{(1)}m^{(2)}+\rho^{(2)}m^{(1)}}=0 \tag{1-16}$$

By the use of approximation of $\mu^{(1)}=\mu^{(2)}$, the third, fourth, fifth and sixth terms of (1-16) are disappeared. The primary solution of (1-16) is obtained by the first and the second term of (1-16).

$$n_0=\pm\sqrt{\frac{k(\rho^{(2)}g+k^2T-\rho^{(1)}g)}{\rho^{(2)}+\rho^{(1)}}}$$

As the motion is limited to progressive wave,

$$n_0=-\sqrt{\frac{k(\rho^{(2)}g+k^2T-\rho^{(1)}g)}{\rho^{(2)}+\rho^{(1)}}} \tag{1-17}$$

By making use of (1-17), (1-16) can be approximated to

$$(\rho^{(2)}g+k^2T-\rho^{(1)}g)-\frac{n^2}{k}(\rho^{(2)}+\rho^{(1)})-\frac{n_0{}^2 4\rho^{(1)}\rho^{(2)}}{\rho^{(1)}m^{(2)}+\rho^{(2)}m^{(1)}}=0 \tag{1-18}$$

The solution of (1-18) is

$$n=n_0\left(1-\frac{2\rho^{(1)}\rho^{(2)}k}{(\rho^{(2)}+\rho^{(1)})(\rho^{(1)}m^{(2)}+\rho^{(2)}m^{(1)})}\right) \tag{1-19}$$

A problem in the expression (1-19) is the estimation of $m^{(1)}$ and $m^{(2)}$, and in our case, if the wave number of the interfacial waves is smaller than about 3 ($k\leqq 3$) (this is the usual case in experiment and in field observation), $m^{(1)}$ and $m^{(2)}$ can be sufficiently approximated by

$$\left.\begin{aligned} m^{(1)}&=\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)\left(\frac{-n_0}{\nu^{(1)}}\right)^{1/2} \\ m^{(2)}&=\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)\left(\frac{-n_0}{\nu^{(2)}}\right)^{1/2} \end{aligned}\right\} \tag{1-20}$$

Using (1-19) and (1-20), (1-10) may be transformed to

$$\left.\begin{aligned} \eta=D\cos&\left[kx+n_0\left\{1-\frac{\sqrt{2}\rho^{(1)}\rho^{(2)}k}{(\rho^{(1)}+\rho^{(2)})(-n_0)^{1/2}\left(\frac{1}{\sqrt{\nu^{(1)}}}+\frac{1}{\sqrt{\nu^{(2)}}}\right)}\right\}t\right] \\ &\cdot\exp\left(-\frac{(-n_0)^{1/2}\sqrt{2}\rho^{(1)}\rho^{(2)}k}{(\rho^{(1)}+\rho^{(2)})\left(\frac{1}{\sqrt{\nu^{(1)}}}+\frac{1}{\sqrt{\nu^{(2)}}}\right)}t\right) \\ \overline{\eta^2}=\frac{D^2}{2}e^{-2\alpha t},&\qquad \alpha=\frac{(-n_0)^{1/2}\sqrt{2}\rho^{(1)}\rho^{(2)}k}{(\rho^{(1)}+\rho^{(2)})\left(\frac{1}{\sqrt{\nu^{(1)}}}+\frac{1}{\sqrt{\nu^{(2)}}}\right)} \end{aligned}\right\} \tag{1-21}$$

The expression of energy of interfacial gravity wave and its attenuation ratio can be expressed by

$$E=\frac{1}{2}(\rho^{(2)}-\rho^{(1)})gD^2e^{-2\alpha t}$$

$$\frac{dE}{dt}=-(\rho^{(2)}-\rho^{(1)})\alpha g D^2 e^{-2\alpha t}=-(-n_0)^{5/2}\frac{\sqrt{2}\mu}{\sqrt{\nu^{(1)}}+\sqrt{\nu^{(2)}}}D'^2 \tag{1-22}$$

In (1–22) the effect of interfacial capillary force is neglected, and $D^2e^{-2\alpha t}$ is expressed by D'^2. The notation μ means the common coefficient of molecular viscosity of both fluids.

In the above computations, we assumed $\mu^{(1)}=\mu^{(2)}$ through fresh and salt water, and this solution can be extended to the case when $\mu^{(1)}$ is a little different from $\mu^{(2)}$ by a simple numerical consideration. At an ordinary difference of temperature of both fluids the correction term is small, and may be neglected in practice.

In the case when both fluids have non-perturbed flow in the horizontal direction, α in (1–21) should be modified by this primary flow. But, as the viscous coefficient of both fluids is finite and stillmore nearly equal, the velocity profile of the primary flow in the present problem cannot jump at the interface and is continuous smoothly. This allows the application of α of (1–21) to the case of flowing water as a practical and effective approximation.

1.3 The case in which the viscous coefficient of the lower fluid can be neglected

In this case the dynamic and kinematic conditions at the interface can be put as follows.

$$-p^{(2)}=T\frac{\partial^2\eta}{\partial x^2}-p^{(1)}+2\mu^{(1)}\frac{\partial v^{(1)}}{\partial y} \qquad \text{at} \quad y=0 \tag{1-23}$$

$$\mu^{(1)}\left(\frac{\partial v^{(1)}}{\partial x}+\frac{\partial u^{(1)}}{\partial y}\right)=0 \qquad \text{at} \quad y=0 \tag{1-24}$$

$$v^{(1)}=v^{(2)}=\frac{\partial\eta}{\partial t} \qquad \text{at} \quad y=0 \tag{1-25}$$

An expression which corresponds to (1–5–2) does not exist. The horizontal velocity at the interface would be discontinuous, because the viscous coefficient is not effective in the lower fluid. We can obtain the characteristic equation in this case as;

$$\left(\rho^{(2)}g\frac{1}{k}-\rho^{(1)}g\frac{1}{k}+\frac{k^2T}{k}\right)-(\rho^{(1)}+\rho^{(2)})\frac{n^2}{k^2}+4\mu^{(1)}k^2\nu^{(1)}$$
$$+i4\rho^{(1)}n\nu^{(1)}-4\mu^{(1)}m^{(1)}k\nu^{(1)}=0 \tag{1-26}$$

The effect of viscid terms seems small, and the lowest approximation of n $(=n_0)$ may be given by the first and second terms of (1–26),

$$n_0=-\sqrt{\frac{k(\rho^{(2)}g-\rho^{(1)}g+k^2T)}{\rho^{(1)}+\rho^{(2)}}} \tag{1-27}$$

The solution which includes the viscous term should be compared with the result of section 1.2, and so we consider the case in which the density of the lower fluid is taken to salt water (its viscosity is neglected), and the density and the coefficient of viscosity of the upper fluid are from fresh water. Equating three viscous terms in (1–26) by making use of these numerical conditions of both fluids, we can find that the third and fifth terms may be discounted at the wave number now considered.

By this way

$$n=n_0\left(1+\frac{i2\mu^{(1)}k^2}{(\rho^{(1)}+\rho^{(2)})n_0}\right) \tag{1-28}$$

$$\left.\begin{aligned}\eta&=D\cos(kx+n_0t)\exp\left(-\frac{2\mu^{(1)}k^2}{\rho^{(1)}+\rho^{(2)}}t\right)\\ \overline{\eta^2}&=\frac{D^2}{2}\exp\left(-\frac{4\mu^{(1)}k^2}{\rho^{(1)}+\rho^{(2)}}t\right)\end{aligned}\right\} \tag{1-29}$$

The expressions of the energy of interfacial waves are;

$$\left.\begin{aligned}E&=\frac{1}{2}(\rho^{(2)}-\rho^{(1)})gD^2\exp\left(-\frac{4\mu^{(1)}k^2}{\rho^{(1)}+\rho^{(2)}}t\right)\\ \frac{dE}{dt}&=(\rho^{(2)}-\rho^{(1)})gD^2\left(-\frac{2\mu^{(1)}k^2}{\rho^{(1)}+\rho^{(2)}}\right)\exp\left(-\frac{4\mu^{(1)}k^2}{\rho^{(1)}+\rho^{(2)}}t\right)\\ &=-2\mu^{(1)}k^3c_0^2D'^2=-2\mu^{(1)}n_0^2kD'^2\end{aligned}\right\} \tag{1-30}$$

Here we put $D^2\cdot\exp\left(-\frac{4\mu^{(1)}k^2}{\rho^{(1)}+\rho^{(2)}}t\right)=D'^2$, and T in (1-27) is neglected.

The previous discussion of the energy dissipation of interfacial waves given by G. H. Keulegan (1949) was based principally on the mechanism shown in this section, and it is clear that we should use the result in section 1.2 at the analytical treatment of interfacial waves of usual estuarial problems.

From (1-22) and (1-30),

$$\left.\begin{aligned}\frac{dE}{dt}&=-(-n_0)^{5/2}\frac{\sqrt{2}\mu}{\sqrt{\nu^{(1)}}+\sqrt{\nu^{(2)}}}D'^2 &&(\text{at}\quad \mu^{(1)}=\mu^{(2)}=\mu)\\ \frac{dE}{dt}&=-2\mu^{(1)}n_0^2kD'^2 &&(\text{at}\quad \mu^{(2)}=0)\end{aligned}\right\} \tag{1-31}$$

and, putting $\frac{dE}{dt}=-\alpha_1D'^2$ and $\frac{dE}{dt}=-\beta_1D'^2$ respectively in the above relations, Table—1 shows the values of α_1, β_1 and α_1/β_1 against k. It is clear that the value of α_1/β_1 is far greater than 1, and the tendency is more distinguished for smaller values of k.

Table—1

k	α_1	β_1	α_1/β_1
5	9.095	4.851	1.874
3	4.801	1.746	2.749
2	2.892	0.7760	3.726
1	1.216	0.1940	6.268
0.5	0.5116	0.04853	10.54
0.1	0.06840	0.001940	35.26
0.05	0.02876	0.0004851	59.29

1.4 Some properties of viscous boundary layer

Computation in section 2 and 3 shows vividly two different cases of boundary layers of interfacial waves. Then the characters of vorticity in these two boundary layers are examined.

The vorticity in both layers is shown by

$$\left.\begin{aligned}\zeta^{(1)}&=\frac{\partial v^{(1)}}{\partial x}-\frac{\partial u^{(1)}}{\partial y}\\ \zeta^{(2)}&=\frac{\partial v^{(2)}}{\partial x}-\frac{\partial u^{(2)}}{\partial y}\end{aligned}\right\}\tag{1-32}$$

In the case of $\mu^{(1)}=\mu^{(2)}$, some computations show that (1-32) is expressed by

$$\left.\begin{aligned}\zeta^{(1)}&=-2nD\frac{m^{(1)}+k}{1+\dfrac{m^{(1)}+k}{m^{(2)}+k}}e^{-m^{(1)}y}e^{i(kx+nt)}\\ \zeta^{(2)}&=-2nD\frac{m^{(2)}+k}{1+\dfrac{m^{(2)}+k}{m^{(1)}+k}}e^{m^{(2)}y}e^{i(kx+nt)}\end{aligned}\right\}\tag{1-33}$$

In the present problem we can use the approximate expression (1-20), and stillmore $m^{(1)}$ is nearly equal to $m^{(2)}$ in practice. Therefore

$$\left.\begin{aligned}\zeta^{(1)}\fallingdotseq\zeta^{(2)}\fallingdotseq&-n_0D\frac{1}{\sqrt{2}}\left(\frac{-n_0}{\nu}\right)^{1/2}\exp\left(\mp\frac{1}{\sqrt{2}}\left(\frac{-n_0}{\nu}\right)^{1/2}y\right)\\ &\times e^{-\alpha t}\cdot\left[\cos\left(\pm\frac{1}{\sqrt{2}}\left(\frac{-n_0}{\nu}\right)^{1/2}y+kx+n_0t\right)\right.\\ &\left.+\sin\left(\pm\frac{1}{\sqrt{2}}\left(\frac{-n_0}{\nu}\right)^{1/2}y+kx+n_0t\right)\right]\\ \alpha\fallingdotseq&\frac{(-n_0)^{1/2}}{2\sqrt{2}}k\sqrt{\nu}\quad\text{(small difference between } \nu^{(1)} \text{ and } \nu^{(2)} \text{ is neglected)}\end{aligned}\right\}\tag{1-34}$$

In the case $\mu^{(2)}=0$,

$$\zeta^{(1)}=-2nkD\cdot e^{-m^{(1)}y}\cdot e^{i(kx+nt)}\,,\quad \zeta^{(2)}=0\tag{1-35}$$

Expressions of (1-33) and (1-35) indicate that the boundary layer is clearly concentrated to the narrow region of the interfacial boundary, and if the lower fluid is inviscid, the vorticity of lower layer disappears, showing that the viscous effect is limited to the upper layer. Because the relation of $|m^{(1)}|\simeq|m^{(2)}|\gg k$ is existent, the strength of vorticity given by (1-33) is far greater than that of (1-35).

The expression of (1-35) is transformed to

$$\left.\begin{aligned}\zeta^{(1)}&\fallingdotseq-2n_0kD\cdot\exp\left(-\frac{1}{\sqrt{2}}\left(\frac{-n_0}{\nu^{(1)}}\right)^{1/2}y\right)\cos\left(\frac{1}{\sqrt{2}}\left(\frac{-n_0}{\nu^{(1)}}\right)^{1/2}y+kx+n_0t\right)e^{-\beta t}\\ &\qquad\beta\fallingdotseq\nu^{(1)}k^2\\ \zeta^{(2)}&=0\end{aligned}\right\}\tag{1-36}$$

In (1-34), putting $-n_0=kc_0$, we take the limit at $y\to 0$. The vorticity at the interfacial boundary is ;

$$\zeta^{(1)}_{y\to 0}\doteqdot\zeta^{(2)}_{y\to 0}\doteqdot kc_0De^{-\alpha t}\cdot\frac{1}{\sqrt{2}}\left(\frac{kc_0}{\nu}\right)^{1/2}\cdot\{\cos(kx+n_0t)+\sin(kx+n_0t)\}$$

Then we use the irrotational horizontal particle velocity just outside the viscous layer to rewrite the above relation.

$$\zeta^{(1)}_{y\to 0}\doteqdot\zeta^{(0)}_{y\to 0}\doteqdot\underset{\substack{\text{inviscid at}\\ y\to 0}}{\begin{Bmatrix}-u^{(1)}\\+u^{(2)}\end{Bmatrix}}\frac{1}{\sqrt{2}}\left(\frac{kc_0}{\nu}\right)^{1/2}\{1+\tan(kx+n_0t)\} \tag{1-37}$$

On the other hand, if we remark on the rigid and smooth boundary layer of the bottom of surface progressive waves, we can obtain the similar form of vorticity with (1-37).

$$\left.\begin{aligned}&\zeta_{y\to 0}=-\underset{\substack{\text{inviscid at}\\ y\to 0}}{u}\cdot\frac{1}{\sqrt{2}}\left(\frac{kc_0}{\nu}\right)^{1/2}\cdot\{1+\tan(kx+n_0t)\}\\ &c_0=\sqrt{\frac{g}{k}\tanh kh}\ ,\qquad -n_0=kc_0\end{aligned}\right\} \tag{1-38}$$

In (1-38) u means the irrotational horizontal particle velocity just above the bottom viscous layer. Stillmore if we examine the boundary layer of steady flow, we can find the very similar expression from the Blasius laminar boundary layer along the flat smooth plate.

In this case the vorticity along the plate is ;

$$\zeta_{y\to 0}\simeq -0.332u_0\sqrt{\frac{u_0}{\nu x}} \tag{1-39}$$

Here u_0 means the velocity of steady flow outside the boundary layer. x is the distance from the tip of the plate. Expressions of (1-37), (1-38) and (1-39) have the common character in their structures, and they are influenced by viscosity by the expression of the same type. If we use the Reynolds number of boundary layer defined by $Re^*=\frac{u\delta^*}{\nu}$, [$u$ means the irrotational periodic horizontal velocity just outside the boundary layer in (1-37) and in (1-38), and δ^* is taken to $\frac{1}{\delta^*}=\frac{1}{\sqrt{2}}\left(\frac{kc_0}{\nu}\right)^{1/2}$. In (1-39) u means the velocity of the steady flow outside the boundary layer, and δ^* is taken to displacement thickness of boundary layer. $\delta^*=1.7208\sqrt{\frac{\nu x}{u_0}}$], from (1-37) for interfacial waves ($\mu^{(1)}=\mu^{(2)}$)

$$\zeta^{(1)}_{y\to 0}\doteqdot\zeta^{(2)}_{y\to 0}\doteqdot -\frac{1}{2}Re^*(-n_0)\{1+\tan(kx+n_0t)\} \tag{1-40}$$

from (1-38) for surface waves

$$\zeta_{y\to 0}=-\frac{1}{2}Re^*(-n_0)\{1+\tan(kx+n_0t)\} \tag{1-41}$$

from (1–39) for Blasius boundary layer

$$\zeta_{y\to 0}=-0.193\,Re^{*}\frac{u_0}{x} \tag{1–42}$$

(in (1–41), n_0 is for surface waves, and u_0/x in (1–42) means the elapsed time of the main flow.) Expressions of (1–40), (1–41) and (1–42) clarify the close and linear relation between the vorticity and the Reynolds number of boundary layer. Re^* is an important factor for the determination of stability criterion of the boundary layer. For large values of Re^*, Blasius boundary layer shows the breakdown of the laminar characteristics indicated by the Tollmien-Schlichting wave (C. C. Lin (1955)). For the bottom boundary layer of surface waves, there is not yet the strict theoretical estimation of stability problem. But some experiment (A. Brebner et al. (1966)) offers certainly some evidence of the breakdown of the laminar layer. Therefore, if we disregard the difference of density of both fluids, the expression of (1–40) for interfacial waves may be sufficient to show the instability of laminar boundary layer at some value of Re^*.

Stillmore in this case the periodic horizontal flow has an inflection point in the thin laminar boundary layer, and from the up-to-date theoretical discussions we know that the critical value of Re^* falls down remarkably by the existence of inflection point in the velocity profile of main flow. In our problem the flow is periodic and is not steady, but Re^* of (1–40) attains $(1\sim5)\times10$ in model experiment, and in usual field study it may be $(1\sim5)\times10^2$. Although the disregard of the difference of density at the interface is a remaind problem, it may be probable that the viscous instability of the interfacial boundary layer is an important factor for the turbulent mixing of salt and fresh water at the boundary.

Then we return to the case of $\mu^{(2)}=0$, and examine (1–36). The expression of (1–36) has the same form of the vorticity of surface free boundary layer of surface waves, which is already shown by H. Lamb (1932) as

$$\zeta=\mp 2k^2Dc_0e^{-2\nu k^2t}\exp\left(\frac{1}{\sqrt{2}}\left(\frac{kc_0}{\nu}\right)^{1/2}y\right)\cdot\cos\left\{kx\pm\left(n_0t+\frac{1}{\sqrt{2}}\left(\frac{kc_0}{\nu}\right)^{1/2}y\right)\right\} \tag{1–43}$$

(In (1–43) $y\leqq0$, c_0 and n_0 are concerned with surface waves.)

In (1–36) and (1–43) ζ at $y\to0$ does not influenced by viscosity except for the attnuation effect of viscosity. The pure free boundary layer of the surface of surface waves seems stable in experiment, and the interfacial boundary in the case $\mu^{(2)}=0$ may be also stable.

The instability problem of the interfacial laminar boundary layer at the case of $\mu^{(1)}\fallingdotseq\mu^{(2)}$ ((1–33) and (1–34) does not need the equality between $\mu^{(1)}$ and $\mu^{(2)}$ strictly in their application.) seems important in the dynamical interpretation of interfacial instability problem of salt and fresh water. Computation in Chapter 2 will show that the permanent type progressive wave of finite height at the interface cannot break down easily when the density of the upper layer is comparable with that of the lower layer. This indicates the stable condition of inviscid wave at the interface of salt and fresh water. (A remark should be made on the condition that the wave length is assumed small compared with the depth of both layers.) Although some doubts may be contained for the inviscid stability at the approximation of higher order in the existence of nonperturbed flow, the instability of viscous boundary layer at the interface may have important roles for the occurence of the turbulent mixing and the additional energy dissipation at the interface.

2. Interfacial waves of finite amplitude (permanent type)

2.1 Basic equations

The problem is limited to the case of irrotational and inviscid. The relations of co-ordinate and notations are same as chapter 1.

$$u^{(1)}=\frac{\partial\phi^{(1)}}{\partial x}, \qquad v^{(1)}=\frac{\partial\phi^{(1)}}{\partial y} \tag{2-1}$$

$$u^{(2)}=\frac{\partial\phi^{(2)}}{\partial x}, \qquad v^{(2)}=\frac{\partial\phi^{(2)}}{\partial y} \tag{2-2}$$

$$\nabla^2\phi^{(1)}=0, \qquad \nabla^2\phi^{(2)}=0 \tag{2-3}$$

The depth of both layers is assumed sufficiently large.

$$\left.\begin{aligned}&\phi^{(1)}\rightarrow \text{a function of } t \text{ only at} \quad y\rightarrow\infty\\ &\phi^{(2)}\rightarrow \text{a function of } t \text{ only at} \quad y\rightarrow-\infty\end{aligned}\right\} \tag{2-4}$$

Integrals of dynamical equations in both layers become

$$\rho^{(1)}\frac{\partial\phi^{(1)}}{\partial t}+\frac{1}{2}\rho^{(1)}q^{(1)2}+\rho^{(1)}gy+p^{(1)}=F^{(1)}(t) \tag{2-5}$$

$$\rho^{(2)}\frac{\partial\phi^{(2)}}{\partial t}+\frac{1}{2}\rho^{(2)}q^{(2)2}+\rho^{(2)}gy+p^{(2)}=F^{(2)}(t) \tag{2-6}$$

When perturbed motions $\phi^{(1)}$ and $\phi^{(2)}$ become negligible, we have at the interface,

$$p_0^{(1)}=F^{(1)}(t)=\text{const.}$$

$$p_0^{(2)}=F^{(2)}(t)=\text{const.}$$

and $p_0^{(1)}=p_0^{(2)}$ is consistent. Therefore $F^{(1)}(t)$ equals $F^{(2)}(t)$ at the interface, and stillmore they are independent of x, y. Putting $p_0^{(1)}=p_0^{(2)}=0$, we can obtain a dynamical condition at the interface;

$$\rho^{(1)}\frac{\partial\phi^{(1)}}{\partial t}+\frac{1}{2}\rho^{(1)}q^{(1)2}+\rho^{(1)}g\eta=\rho^{(2)}\frac{\partial\phi^{(2)}}{\partial t}+\frac{1}{2}\rho^{(2)}q^{(2)2}+\rho^{2}g\eta \quad \text{at} \quad y=\eta \tag{2-7}$$

Kinematic conditions at the interface are;

$$\frac{\partial\eta}{\partial t}+u^{(1)}\frac{\partial\eta}{\partial x}=v^{(1)} \qquad \text{at} \quad y=\eta \tag{2-8}$$

$$\frac{\partial\eta}{\partial t}+u^{(2)}\frac{\partial\eta}{\partial x}=v^{(2)} \qquad \text{at} \quad y=\eta \tag{2-9}$$

From the condition of permanent type;

$$\left.\begin{aligned}&\phi^{(1)}=\phi^{(1)}(x-ct)\\&\phi^{(2)}=\phi^{(2)}(x-ct)\\&\eta=\eta(x-ct)\end{aligned}\right\}\tag{2-10}$$

From (2-10)

$$\left.\begin{aligned}&\frac{\partial\phi^{(1)}}{\partial t}=-c\frac{\partial\phi^{(1)}}{\partial x}\\&\frac{\partial\phi^{(2)}}{\partial t}=-c\frac{\partial\phi^{(2)}}{\partial x}\\&\frac{\partial\eta}{\partial t}=-c\frac{\partial\eta}{\partial x}\end{aligned}\right\}\tag{2-11}$$

From (2-7), (2-8), (2-9) and (2-11)

$$-\rho^{(1)}c\frac{\partial\phi^{(1)}}{\partial x}+\frac{1}{2}\rho^{(1)}\left\{\left(\frac{\partial\phi^{(1)}}{\partial x}\right)^2+\left(\frac{\partial\phi^{(1)}}{\partial y}\right)^2\right\}+\rho^{(1)}g\eta$$

$$=-\rho^{(2)}c\frac{\partial\phi^{(2)}}{\partial x}+\frac{1}{2}\rho^{(2)}\left\{\left(\frac{\partial\phi^{(2)}}{\partial x}\right)^2+\left(\frac{\partial\phi^{(2)}}{\partial y}\right)^2\right\}+\rho^{(2)}g\eta \quad \text{at} \quad y=\eta \tag{2-12}$$

$$-c\frac{\partial\eta}{\partial x}+\frac{\partial\phi^{(1)}}{\partial x}\frac{\partial\eta}{\partial x}=\frac{\partial\phi^{(1)}}{\partial y} \quad \text{at} \quad y=\eta \tag{2-13}$$

$$-c\frac{\partial\eta}{\partial x}+\frac{\partial\phi^{(2)}}{\partial x}\frac{\partial\eta}{\partial x}=\frac{\partial\phi^{(2)}}{\partial y} \quad \text{at} \quad y=\eta \tag{2-14}$$

The slope of wave profile at the interface is used as the parametre of perturbation ;

$$\left.\begin{aligned}&\phi^{(1)}=\alpha\phi_1{}^{(1)}+\alpha^2\phi_2{}^{(1)}+\alpha^3\phi_3{}^{(1)}+\cdots\cdots\\&\phi^{(2)}=\alpha\phi_1{}^{(2)}+\alpha^2\phi_2{}^{(2)}+\alpha^3\phi_3{}^{(2)}+\cdots\cdots\\&\eta=\alpha\eta_1+\alpha^2\eta_2+\alpha^3\eta_3+\cdots\cdots\\&c=c_0+\alpha c_1+\alpha^2c_2+\alpha^3c_3+\cdots\cdots\end{aligned}\right\}\tag{2-15}$$

Interfacial conditions (2-12), (2-13) and (2-14) are perturbed by (2-15), and conditions at $y=\eta$ are reduced to conditions at $y=0$ by making use of the Taylor's expansion around $y=0$. From (2-12),

$$-\rho^{(1)}c_0\frac{\partial\phi_1{}^{(1)}(0)}{\partial x}+\rho^{(1)}g\eta_1=-\rho^{(2)}c_0\frac{\partial\phi_1{}^{(2)}(0)}{\partial x}+\rho^{(2)}g\eta_1 \quad \text{at} \quad y=0 \tag{2-16-1}$$

$$-\rho^{(1)}c_0\frac{\partial\phi_2{}^{(1)}(0)}{\partial x}-\rho^{(1)}c_0\frac{\partial^2\phi_1{}^{(1)}(0)}{\partial x\,\partial y}\eta_1-\rho^{(1)}c_1\frac{\partial\phi_1{}^{(1)}(0)}{\partial x}$$

$$+\frac{1}{2}\rho^{(1)}\left(\frac{\partial\phi_1{}^{(1)}(0)}{\partial x}\right)^2+\frac{1}{2}\rho^{(1)}\left(\frac{\partial\phi_1{}^{(1)}(0)}{\partial y}\right)^2+\rho^{(1)}g\eta_2$$

$$=-\rho^{(2)}c_0\frac{\partial\phi_2{}^{(2)}(0)}{\partial x}-\rho^{(2)}c_0\frac{\partial^2\phi_1{}^{(2)}(0)}{\partial x\,\partial y}\eta_1-\rho^{(2)}c_1\frac{\partial\phi_1{}^{(2)}(0)}{\partial x}$$

$$+\frac{1}{2}\rho^{(2)}\left(\frac{\partial\phi_1{}^{(2)}(0)}{\partial x}\right)^2+\frac{1}{2}\rho^{(2)}\left(\frac{\partial\phi_1{}^{(2)}(0)}{\partial y}\right)^2+\rho^{(2)}g\eta_2 \quad \text{at} \quad y=0 \tag{2-16-2}$$

$$-\rho^{(1)}c_0\frac{\partial\phi_3^{(1)}(0)}{\partial x}-\rho^{(1)}c_0\frac{\partial^2\phi_1^{(1)}(0)}{\partial x\,\partial y}\eta_2-\rho^{(1)}c_0\frac{\partial^2\phi_2^{(1)}(0)}{\partial x\,\partial y}\eta_1$$

$$-\frac{1}{2}\rho^{(1)}c_0\frac{\partial^3\phi_1^{(1)}(0)}{\partial x\,\partial y^2}\eta_1^2-\rho^{(1)}c_1\frac{\partial\phi_2^{(1)}(0)}{\partial x}-\rho^{(1)}c_1\frac{\partial^2\phi_1^{(1)}(0)}{\partial x\,\partial y}\eta_1$$

$$-\rho^{(1)}c_2\frac{\partial\phi_1^{(1)}(0)}{\partial x}+\rho^{(1)}\frac{\partial\phi_1^{(1)}(0)}{\partial x}\frac{\partial\phi_2^{(1)}(0)}{\partial x}+\rho^{(1)}\frac{\partial\phi_1^{(1)}(0)}{\partial y}\frac{\partial\phi_2^{(1)}(0)}{\partial y}$$

$$+\rho^{(1)}\frac{\partial\phi_1^{(1)}(0)}{\partial x}\frac{\partial^2\phi_1^{(1)}(0)}{\partial x\,\partial y}\eta_1+\rho^{(1)}\frac{\partial\phi_1^{(1)}(0)}{\partial y}\frac{\partial^2\phi_1^{(1)}(0)}{\partial y^2}\eta_1+\rho^{(1)}g\eta_3$$

$$=-\rho^{(2)}c_0\frac{\partial\phi_3^{(2)}(0)}{\partial x}-\rho^{(2)}c_0\frac{\partial^2\phi_1^{(2)}(0)}{\partial x\,\partial y}\eta_2-\rho^{(2)}c_0\frac{\partial^2\phi_2^{(2)}(0)}{\partial x\,\partial y}\eta_1$$

$$-\frac{1}{2}\rho^{(2)}c_0\frac{\partial^3\phi_1^{(2)}(0)}{\partial x\,\partial y^2}\eta_1^2-\rho^{(2)}c_1\frac{\partial\phi_2^{(2)}(0)}{\partial x}-\rho^{(2)}c_1\frac{\partial^2\phi_1^{(2)}(0)}{\partial x\,\partial y}\eta_1$$

$$-\rho^{(2)}c_2\frac{\partial\phi_1^{(2)}(0)}{\partial x}+\rho^{(2)}\frac{\partial\phi_1^{(2)}(0)}{\partial x}\frac{\partial\phi_2^{(2)}(0)}{\partial x}+\rho^{(2)}\frac{\partial\phi_1^{(2)}(0)}{\partial y}\frac{\partial\phi_2^{(2)}(0)}{\partial y}$$

$$+\rho^{(2)}\frac{\partial\phi_1^{(2)}(0)}{\partial x}\frac{\partial^2\phi_1^{(2)}(0)}{\partial x\,\partial y}\eta_1+\rho^{(2)}\frac{\partial\phi_1^{(2)}(0)}{\partial y}\frac{\partial^2\phi_1^{(2)}(0)}{\partial y^2}\eta_1+\rho^{(2)}g\eta_3 \quad \text{at} \quad y=0 \tag{2-16-3}$$

From (2–13),

$$-c_0\frac{\partial\eta_1}{\partial x}=\frac{\partial\phi_1^{(1)}(0)}{\partial y} \quad \text{at} \quad y=0 \tag{2-17-1}$$

$$-c_0\frac{\partial\eta_2}{\partial x}-c_1\frac{\partial\eta_1}{\partial x}+\frac{\partial\phi_1^{(1)}(0)}{\partial x}\frac{\partial\eta_1}{\partial x}=\frac{\partial\phi_2^{(1)}(0)}{\partial y}+\frac{\partial^2\phi_1^{(1)}(0)}{\partial y^2}\eta_1 \quad \text{at} \quad y=0 \tag{2-17-2}$$

$$-c_0\frac{\partial\eta_3}{\partial x}-c_1\frac{\partial\eta_2}{\partial x}-c_2\frac{\partial\eta_1}{\partial x}+\frac{\partial\phi_2^{(1)}(0)}{\partial x}\frac{\partial\eta_1}{\partial x}+\frac{\partial^2\phi_1^{(1)}(0)}{\partial x\,\partial y}\frac{\partial\eta_1}{\partial x}\eta_1+\frac{\partial\phi_1^{(1)}(0)}{\partial x}\frac{\partial\eta_2}{\partial x}$$

$$=\frac{\partial\phi_3^{(1)}(0)}{\partial y}+\frac{\partial^2\phi_1^{(1)}(0)}{\partial y^2}\eta_2+\frac{\partial^2\phi_2^{(1)}(0)}{\partial y^2}\eta_1+\frac{1}{2}\frac{\partial^3\phi_1^{(1)}(0)}{\partial y^3}\eta_1^2 \quad \text{at} \quad y=0 \tag{2-17-3}$$

From (2–14),

$$-c_0\frac{\partial\eta_1}{\partial x}=\frac{\partial\phi_1^{(2)}(0)}{\partial y} \quad \text{at} \quad y=0 \tag{2-18-1}$$

$$-c_0\frac{\partial\eta_2}{\partial x}-c_1\frac{\partial\eta_1}{\partial x}+\frac{\partial\phi_1^{(2)}(0)}{\partial x}\frac{\partial\eta_1}{\partial x}=\frac{\partial\phi_2^{(2)}(0)}{\partial y}+\frac{\partial^2\phi_1^{(2)}(0)}{\partial y^2}\eta_1 \quad \text{at} \quad y=0 \tag{2-18-2}$$

$$-c_0\frac{\partial\eta_3}{\partial x}-c_1\frac{\partial\eta_2}{\partial x}-c_2\frac{\partial\eta_1}{\partial x}+\frac{\partial\phi_2^{(2)}(0)}{\partial x}\frac{\partial\eta_1}{\partial x}+\frac{\partial^2\phi_1^{(2)}(0)}{\partial x\,\partial y}\eta_1\frac{\partial\eta_1}{\partial x}+\frac{\partial\phi_1^{(2)}(0)}{\partial x}\frac{\partial\eta_2}{\partial x}$$

$$=\frac{\partial\phi_3^{(2)}(0)}{\partial y}+\frac{\partial^2\phi_1^{(2)}(0)}{\partial y^2}\eta_2+\frac{\partial^2\phi_2^{(2)}(0)}{\partial y^2}\eta_1+\frac{1}{2}\frac{\partial^3\phi_1^{(2)}(0)}{\partial y^3}\eta_1^2 \quad \text{at} \quad y=0 \tag{2-18-3}$$

Stillmore, from the equation of continuity,

$$\left.\begin{array}{lll}\nabla^2\phi_1^{(1)}=0\,, & \nabla^2\phi_2^{(1)}=0\,, & \nabla^2\phi_3^{(1)}=0\\ \nabla^2\phi_1^{(2)}=0\,, & \nabla^2\phi_2^{(2)}=0\,, & \nabla^2\phi_3^{(2)}=0\end{array}\right\} \tag{2-19}$$

Relations of (2–16), (2–17), (2–18) and (2–19) are used to obtain the perturbed solutions of this problem.

2.2 Approximations of the first, second and third order

The first and second order approximations are determined by making use of the similar method with the perturbed solutions of surface waves.

The first order approximation is ;

$$\eta_1 = A_1 \cos k(x-ct) \qquad (A_1 : \text{real and positive}) \tag{2-20}$$

$$\left.\begin{aligned} \phi_1{}^{(1)} &= -c_0 A_1 e^{-ky} \cdot \sin k(x-ct) \\ \phi_1{}^{(2)} &= c_0 A_1 e^{ky} \cdot \sin k(x-ct) \end{aligned}\right\} \tag{2-21}$$

$$c_0 = \sqrt{\frac{g}{k} \frac{\rho^{(2)} - \rho^{(1)}}{\rho^{(2)} + \rho^{(1)}}} \qquad (c_0 > 0) \tag{2-22}$$

The second order approximation is ;

$$\eta_2 = \frac{1}{2} A_1{}^2 k \frac{\rho^{(2)} - \rho^{(1)}}{\rho^{(2)} + \rho^{(1)}} \cdot \cos 2k(x-ct) \tag{2-23}$$

$$\left.\begin{aligned} \phi_2{}^{(1)} &= -c_0 A_1{}^2 k \frac{\rho^{(2)}}{\rho^{(2)} + \rho^{(1)}} e^{-2ky} \cdot \sin 2k(x-ct) \\ \phi_2{}^{(2)} &= -c_0 A_1{}^2 k \frac{\rho^{(1)}}{\rho^{(2)} + \rho^{(1)}} e^{2ky} \cdot \sin 2k(x-ct) \end{aligned}\right\} \tag{2-24}$$

$$c_1 = 0 \tag{2-25}$$

In the computation of the third order approximation, we use the notations as follows in (2–23) and (2–24).

$$\left.\begin{aligned} \frac{1}{2} A_1{}^2 k \frac{\rho^{(2)} - \rho^{(1)}}{\rho^{(2)} + \rho^{(1)}} &= A_{22} \\ -c_0 A_1{}^2 k \frac{\rho^{(2)}}{\rho^{(2)} + \rho^{(1)}} &= B_{22}^{(1)} \\ -c_0 A_1{}^2 k \frac{\rho^{(1)}}{\rho^{(2)} + \rho^{(1)}} &= B_{22}^{(2)} \end{aligned}\right\} \tag{2-26}$$

In the present problem, the third order approximation may be assumed as ;

$$\left.\begin{aligned} \eta_3 &= A_{31} \cos k(x-ct) + A_{33} \cos 3k(x-ct) \\ \phi_3{}^{(1)} &= B_{31}^{(1)} e^{-ky} \cdot \sin k(x-ct) + B_{33}^{(1)} e^{-3ky} \cdot \sin 3k(x-ct) \\ \phi_3{}^{(2)} &= B_{31}^{(2)} e^{ky} \cdot \sin k(x-ct) + B_{33}^{(2)} e^{3ky} \cdot \sin 3k(x-ct) \end{aligned}\right\} \tag{2-27}$$

From (2–17–3)

$$B_{31}^{(1)} = -c_0 A_{31} - c_2 A_1 + \frac{3}{8} c_0 A_1{}^3 k^2 - c_0 k A_1 \frac{A_{22}}{2} + k A_1 B_{22}^{(1)} \tag{2-28}$$

$$B_{33}^{(1)} = -c_0 A_{33} + A_1 k B_{22}^{(1)} + \frac{1}{8} c_0 A_1{}^3 k^2 - \frac{1}{2} c_0 k A_1 A_{22} \tag{2-29}$$

From (2-18-3)

$$B_{31}^{(2)}=c_0A_{31}+c_2A_1-A_1kB_{22}^{(2)}-\frac{3}{8}c_0A_1^3k^2-\frac{1}{2}c_0kA_1A_{22} \tag{2-30}$$

$$B_{33}^{(2)}=c_0A_{33}-A_1kB_{22}^{(2)}-\frac{1}{8}c_0A_1^3k^2-\frac{1}{2}c_0kA_1A_{22} \tag{2-31}$$

A_{31}, $B_{31}^{(1)}$, $B_{31}^{(2)}$ and c_2 should be determined by (2-28) and (2-30), and A_{33}, $B_{33}^{(1)}$ and $B_{33}^{(2)}$ should be given by (2-29) and (2-31). One more relation for these computations is (2-16-3) (the dynamical condition of interfacial boundary). From this we have two expressions.

The first one is

$$\rho^{(1)}c_0^2kA_{31}+2\rho^{(1)}c_0c_2kA_1-\rho^{(1)}c_0k^2A_1B_{22}^{(1)}-\rho^{(1)}c_0^2A_1^3k^3+\rho^{(1)}gA_{31}$$

$$=-\rho^{(2)}c_0^2kA_{31}-2\rho^{(2)}c_0c_2kA_1+\rho^{(2)}c_0k^2A_1B_{22}^{(2)}+\rho^{(2)}c_0^2A_1^3k^3+\rho^{(2)}gA_{31} \tag{2-32}$$

If we use a first order relation $(\rho^{(1)}+\rho^{(2)})c_0^2k=(\rho^{(2)}-\rho^{(1)})g$ in (2-32), c_2 is determined as

$$c_2=\frac{1}{2}c_0A_1^2k^2\frac{\rho^{(1)2}+\rho^{(2)2}}{(\rho^{(1)}+\rho^{(2)})^2} \tag{2-33}$$

But A_{31}, $B_{31}^{(1)}$ and $B_{31}^{(2)}$ cannot be determined unanimously. If we use a similar method with that of surface waves, the following two cases may be obtained as the representative solutions.

(i) the case of $B_{31}^{(2)}=0$

$$A_{31}=\frac{A_1^3k^2}{8(\rho^{(1)}+\rho^{(2)})^2}\cdot(\rho^{(2)2}-2\rho^{(1)}\rho^{(2)}-11\rho^{(1)2}) \tag{2-34}$$

$$B_{31}^{(1)}=-\frac{3}{2}c_0A_1^3k^2\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}} \tag{2-35}$$

(ii) the case of $A_{31}=0$

$$B_{31}^{(1)}=\frac{1}{8}c_0A_1^3k^2\frac{\rho^{(1)2}-2\rho^{(1)}\rho^{(2)}-11\rho^{(2)2}}{(\rho^{(1)}+\rho^{(2)})^2} \tag{2-36}$$

$$B_{31}^{(2)}=\frac{1}{8}c_0A_1^3k^2\frac{11\rho^{(1)2}+2\rho^{(1)}\rho^{(2)}-\rho^{(2)2}}{(\rho^{(1)}+\rho^{(2)})^2} \tag{2-37}$$

The second relation is

$$-\rho^{(1)}c_0 3kB_{33}^{(1)}-\rho^{(1)}\frac{1}{2}c_0^2A_1k^2A_{22}$$

$$+\rho^{(1)}c_0 2k^2B_{22}^{(1)}A_1+\frac{1}{8}\rho^{(1)}c_0^2A_1^3k^3+\rho^{(1)}gA_{33}$$

$$=-\rho^{(2)}c_0 3kB_{33}^{(2)}-\rho^{(2)}\frac{1}{2}c_0^2k^2A_1A_{22}$$

$$-\rho^{(2)}c_0 2k^2B_{22}^{(2)}A_1-\frac{1}{8}\rho^{(2)}c_0^2A_1^3k^3+\rho^{(2)}gA_{33} \tag{2-38}$$

A_{33}, $B_{33}^{(1)}$ and $B_{33}^{(2)}$ can be determined without objection by (2-29), (2-31) and (2-38).

$$A_{33}=\frac{k^2A_1^3}{4}\left\{\frac{3}{2}-\frac{8\rho^{(1)}\rho^{(2)}}{(\rho^{(1)}+\rho^{(2)})^2}\right\} \tag{2-39}$$

$$B_{33}^{(1)}=-c_0\frac{k^2A_1^3}{4}\left\{1+\frac{5\rho^{(2)}}{\rho^{(2)}+\rho^{(1)}}-\frac{\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}-\frac{8\rho^{(1)}\rho^{(2)}}{(\rho^{(1)}+\rho^{(2)})^2}\right\} \tag{2-40}$$

$$B_{33}^{(2)}=c_0\frac{k^2A_1^3}{4}\left\{1+\frac{5\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}-\frac{\rho^{(2)}}{\rho^{(2)}+\rho^{(1)}}-\frac{8\rho^{(1)}\rho^{(2)}}{(\rho^{(1)}+\rho^{(2)})^2}\right\} \tag{2-41}$$

By this way the solution expanded to the third order approximation may be expressed by

(i) the case of $B_{31}^{(2)}=0$,

$$\begin{aligned}\eta=&A_1\cos k(x-ct)+\frac{1}{2}A_1^2k\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}\cos 2k(x-ct)\\&+\frac{1}{8}\frac{A_1^3k^2}{(\rho^{(1)}+\rho^{(2)})^2}\{\rho^{(2)2}-2\rho^{(1)}\rho^{(2)}-11\rho^{(1)2}\}\cos k(x-ct)\\&+\frac{1}{4}A_1^3k^2\left\{\frac{3}{2}-\frac{8\rho^{(1)}\rho^{(2)}}{(\rho^{(1)}+\rho^{(2)})^2}\right\}\cos 3k(x-ct)\end{aligned} \tag{2-42}$$

$$\begin{aligned}\phi^{(1)}=&-c_0A_1e^{-ky}\sin k(x-ct)-c_0A_1^2k\frac{\rho^{(2)}}{\rho^{(2)}+\rho^{(1)}}e^{-2ky}\cdot\sin 2k(x-ct)\\&-\frac{3}{2}c_0A_1^3k^2\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}e^{-ky}\cdot\sin k(x-ct)\\&-\frac{1}{4}c_0A_1^3k^2\left\{1+\frac{5\rho^{(2)}}{\rho^{(2)}+\rho^{(1)}}-\frac{\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}-\frac{8\rho^{(1)}\rho^{(2)}}{(\rho^{(1)}+\rho^{(2)})^2}\right\}e^{-3ky}\cdot\sin 3k(x-ct)\end{aligned} \tag{2-43}$$

$$\begin{aligned}\phi^{(2)}=&c_0A_1e^{ky}\sin k(x-ct)-c_0A_1^2k\frac{\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}e^{2ky}\cdot\sin 2k(x-ct)\\&+\frac{1}{4}c_0A_1^3k^2\left\{1+\frac{5\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}-\frac{\rho^{(2)}}{\rho^{(2)}+\rho^{(1)}}-\frac{8\rho^{(1)}\rho^{(2)}}{(\rho^{(1)}+\rho^{(2)})^2}\right\}e^{3ky}\cdot\sin 3k(x-ct)\end{aligned} \tag{2-44}$$

(ii) the case of $A_{31}=0$,

$$\begin{aligned}\eta=&A_1\cos k(x-ct)+\frac{1}{2}A_1^2k\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}\cdot\cos 2k(x-ct)\\&+\frac{1}{4}A_1^3k^2\left\{\frac{3}{2}-\frac{8\rho^{(1)}\rho^{(2)}}{(\rho^{(1)}+\rho^{(2)})^2}\right\}\cos 3k(x-ct)\end{aligned} \tag{2-45}$$

$$\begin{aligned}\phi^{(1)}=&-c_0A_1e^{-ky}\sin k(x-ct)-c_0A_1^2k\frac{\rho^{(2)}}{\rho^{(2)}+\rho^{(1)}}e^{-2ky}\cdot\sin 2k(x-ct)\\&+\frac{1}{8}c_0A_1^3k^2\frac{\rho^{(1)2}-2\rho^{(1)}\rho^{(2)}-11\rho^{(2)2}}{(\rho^{(1)}+\rho^{(2)})^2}e^{-ky}\cdot\sin k(x-ct)\\&-\frac{1}{4}c_0A_1^3k^2\left\{1+\frac{5\rho^{(2)}}{\rho^{(2)}+\rho^{(1)}}-\frac{\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}-\frac{8\rho^{(1)}\rho^{(2)}}{(\rho^{(1)}+\rho^{(2)})^2}\right\}e^{-3ky}\cdot\sin 3k(x-ct)\end{aligned} \tag{2-46}$$

$$\phi^{(2)}=c_0A_1e^{ky}\cdot\sin k(x-ct)-c_0A_1^2k\frac{\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}e^{2ky}\cdot\sin 2k(x-ct)$$
$$+\frac{1}{8}c_0A_1^3k^2\frac{11\rho^{(1)2}+2\rho^{(1)}\rho^{(2)}-\rho^{(2)2}}{(\rho^{(1)}+\rho^{(2)})^2}e^{ky}\cdot\sin k(x-ct)$$
$$+\frac{1}{4}c_0A_1^3k^2\left\{1+\frac{5\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}-\frac{\rho^{(2)}}{\rho^{(2)}+\rho^{(1)}}-\frac{8\rho^{(1)}\rho^{(2)}}{(\rho^{(1)}+\rho^{(2)})^2}\right\}e^{3ky}\cdot\sin 3k(x-ct) \tag{2-47}$$

c is common in both (i) and (ii).

$$c=c_0\left\{1+\frac{1}{2}A_1^2k^2\frac{\rho^{(1)2}+\rho^{(2)2}}{(\rho^{(1)}+\rho^{(2)})^2}\right\},\qquad c_0=\sqrt{\frac{g}{k}\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}} \tag{2-48}$$

Both cases of (i) and (ii) given here may be two representative solutions of this problem, but they are influenced by an uncertain selection at the third order approximation.

If we compare these solutions with those of surface waves, the expressions of (2-42), (2-43) and (2-44) are in agreement with E. V. Laitone ((1962) P. 1561 relations (32) and (33)) at $\rho^{(1)}\to 0$, and relations of (2-45), (2-46) and (2-47) correspond with O. M. Phillips ((1960) P. 210 relations (5.9), (5.10) and the one under them) at $\rho^{(1)}\to 0$. The difference of wave profile (η) is not remarkable at $\rho^{(1)}/\rho^{(2)}\to 0$ in both solutions, but at $\rho^{(1)}/\rho^{(2)}\to 1$ it is not so small. Accordingly if we treat the profile of interfacial wave at the interface between fresh and salt water, this uncertain property of the third order approximation can not be negligible in practice.

Some additional remark: (a) At $\rho^{(1)}/\rho^{(2)}\to 1$, η_3 has the negative sign when $(x-ct)$ is equal to zero. This is opposite with the case of surface wave ($\rho^{(1)}\to 0$), and stillmore at $\rho^{(1)}/\rho^{(2)}\to 1$, η_2 approaches zero. So the wave profile at $\rho^{(1)}/\rho^{(2)}\to 1$ is very different from the case of usual surface wave. (b) The experimental verification of (2-48) seems very difficult, because as shown by (1-21) viscous effect on the wave celerity is not so small at the interfacial wave of salt and fresh water, and the existence of the thin mixed layer between both fluids will decrease the celerity of the wave of same wave number.

2.3 The problem of breaking of interfacial waves

This problem seems a complicated one, and the deformation of wave profile caused by the high order process (an example will be referred in chapter 3) should be included. But here we treat the problem in its simplest form. The result will show that the interfacial wave at the interface of fresh and salt water cannot break at the condition of permanent type.

As the approximate solution in this chapter is limited within the third order, it is not sufficient to determine the strict value of wave steepness at the breaking even in a case of surface wave. But, when $\rho^{(1)}$ approaches to $\rho^{(2)}$, a problem remains in which the procedure of perturbation used here can be available to the analysis of breaking. Therefore the method used here is only applicable to the estimation of breaking problem, in which the small value of $\rho^{(1)}/\rho^{(2)}$ is allowed. So the present estimation is qualitative one, and does not give a decisive conception.

From (2-42), (2-43) and (2-44) $u^{(2)}$ at the interfacial boundary $(y=\eta)$ may be determined within the range of the third order approximation;

$$\underset{\text{at } y=\eta}{u^{(2)}}=\frac{1}{2}c_0A_1{}^2k^2+c_0A_1k\left\{1+\frac{1}{4}A_1{}^2k^2\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}\right.$$
$$\left.+\frac{3}{8}A_1{}^2k^2-2A_1{}^2k^2\frac{\rho^{(1)}}{\rho^{(1)}+\rho^{(2)}}\right\}\cos k(x-ct)$$
$$+c_0A_1{}^2k^2\left\{\frac{1}{2}-\frac{2\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}\right\}\cos 2k(x-ct)$$
$$+c_0A_1{}^3k^3\left\{\frac{7}{8}+\frac{3}{2}\frac{\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}-\frac{1}{2}\frac{\rho^{(2)}}{\rho^{(2)}+\rho^{(1)}}\right.$$
$$\left.-\frac{6\rho^{(1)}\rho^{(2)}}{(\rho^{(1)}+\rho^{(2)})^2}\right\}\cos 3k(x-ct) \qquad (2\text{-}49)$$

If $\underset{\text{at } y=\eta}{u^{(2)}}$ of (2-49) is equal to c of (2-48) at $x-ct=0$, it may be considered as the limit of stability of waves in the present approximation. By this way we can estimate the approximate value of A_1k at the limit of stability.

At $\rho^{(1)}=0.000$, $\rho^{(2)}=1.020$

$$(A_1k)^3+0.5(A_1k)^2+A_1k-1=0 \qquad (2\text{-}50\text{-}1)$$

At $\rho^{(1)}=0.200$, $\rho^{(2)}=1.020$

$$0.0954(A_1k)^3+0.3092(A_1k)^2+A_1k-1=0 \qquad (2\text{-}50\text{-}2)$$

At $\rho^{(1)}=0.400$, $\rho^{(2)}=1.020$

$$-0.355(A_1k)^3+0.139(A_1k)^2+A_1k-1=0 \qquad (2\text{-}50\text{-}3)$$

Here we must recall the fact that the value of A_1k should be smaller than 1 in the present method of perturbation, and we seek the value of A_1k which satisfies (2-50-1, 2, 3) in a condition $1>A_1k>0$. This shows $A_1k=0.602$ in (2-50-1) and $A_1k=0.775$ in (2-50-2). But in (2-50-3) we cannot obtain the proper value of A_1k, and the procedure shows that the value of A_1k at the stability limit increases with the increase of $\rho^{(1)}$ in (2-50-1, 2, 3).

we can also compute $\underset{\text{at } y=\eta}{u^{(2)}}$ from (2-45), (2-46) and (2-47) as same as (2-49).

$$\underset{\text{at } y=\eta}{u^{(2)}}=\frac{1}{2}c_0A_1{}^2k^2+c_0A_1k\left\{1+\frac{1}{4}A_1{}^2k^2\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}+\frac{3}{8}A_1{}^2k^2-2A_1{}^2k^2\frac{\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}\right.$$
$$\left.+\frac{1}{8}A_1{}^2k^2\frac{11\rho^{(1)2}+2\rho^{(1)}\rho^{(2)}-\rho^{(2)2}}{(\rho^{(1)}+\rho^{(2)})^2}\right\}\cos k(x-ct)$$
$$+c_0A_1{}^2k^2\left\{\frac{1}{2}-\frac{2\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}\right\}\cos 2k(x-ct)$$
$$+c_0A_1{}^3k^3\left\{\frac{7}{8}+\frac{3}{2}\frac{\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}\right.$$
$$\left.-\frac{1}{2}\frac{\rho^{(2)}}{\rho^{(2)}+\rho^{(1)}}-\frac{6\rho^{(1)}\rho^{(2)}}{(\rho^{(1)}+\rho^{(2)})^2}\right\}\cos 3k(x-ct) \qquad (2\text{-}51)$$

From (2-51) and (2-48) A_1k at the limit of the stability of waves is given by;

At $\rho^{(1)}=0.000$, $\rho^{(2)}=1.020$

$$0.875(A_1k)^3+0.5(A_1k)^2+A_1k-1=0 \tag{2-52-1}$$

At $\rho^{(1)}=0.200$, $\rho^{(2)}=1.020$

$$0.0793(A_1k)^3+0.309(A_1k)^2+A_1k-1=0 \tag{2-52-2}$$

At $\rho^{(1)}=0.400$, $\rho^{(2)}=1.020$

$$-0.259(A_1k)^3+0.139(A_1k)^2+A_1k-1=0 \tag{2-52-3}$$

We obtain the proper values of A_1k from (2-52-1, 2) $\{A_1k=0.612$ in (2-52-1), $A_1k=0.775$ in (2-52-2).$\}$ But we cannot have the proper value of A_1k from (2-52-3), and we see that the tendency is quite same in both cases of (2-49) and (2-51). Therefore we can say that the value of A_1k (wave steepness) at the limit of stability increases with the increase of $\rho^{(1)}$, and that in actual problems of interfacial waves the existence of maximum value of A_1k seems doubtful at $\rho^{(1)} \to \rho^{(2)}$ in the condition of inviscid permanent type.

The more rigorous estimation of this problem may be given as follows. We use the dynamical condition of interface (2-12). J. N. Hunt (1961) transformed (2-12) into the following form, in which he made a motion steady using the uniform opposing current with the velocity $(-c)$.

$$\frac{g}{kc^2}(\rho^{(2)}-\rho^{(1)})e^{-\tau}\cdot\sin\theta=\rho^{(2)}e^{2\tau}\frac{d\tau}{d\sigma}+\rho^{(1)}e^{-2\tau}\frac{d\tau}{d\sigma} \tag{2-53}$$

Physical meanings of θ, τ and σ are shown in Hunt's paper. In (2-53) $\theta \to 0$ and $\tau \to (-)\infty$ are consistent at the limit in which the particle velocity of the crest of waves of lower fluid approaches the wave celerity boundlessly. On the other hand there is a condition;

$$u^{(2)}\cdot dx+v^{(2)}\cdot dy=-\frac{c}{k}d\sigma$$

(in this expression $u^{(2)}$ means the value $\{u^{(2)}$ in (2-2)$-c\}$)
and so $u^{(2)} \to 0$, $v^{(2)} \to 0$ at the limit of instability means $d\sigma \to 0$. At the same time $\tau \to (-)\infty$ indicates that $d\tau$ is finite. By making use of these conditions we estimate each term of (2-53).

$$\left.\begin{array}{l}\text{the left hand side} \longrightarrow \infty\times 0\\ \text{the first term of the right hand side} \longrightarrow 0\times\infty\\ \text{the second term of the right hand side} \longrightarrow \infty\times\infty\end{array}\right\} \tag{2-54}$$

(2-54) indicates that the order of the left hand side of (2-53) is in agreement with the first term of the right hand side, and that the second term of the right hand side is far greater than the other two terms. This inconsistency appeared in (2-53) may be interpreted as the impossibility of breaking of waves under the present circumstance. If $\rho^{(1)}=0$ is satisfied, both sides of (2-53) are of same order at the extreme condition, and the breaking of waves is apparently possible.

The result of this analysis is in agreement with the approximate treatment of (2-50-1, 2, 3) and (2-52-1, 2, 3), qualitatively. The discussion at the end of chapter 1, in which we noticed the importance of instability of viscous boundary layer for the turbulent mixing of superposed fresh and salt water, depends on this tendency of interfacial waves.

3. The Kelvin-Helmholtz instability

3.1 The method of perturbation and the linear relations

The Kelvin-Helmholtz instability is an old treatment of the instability of interface, and it contains some fundamental properties of the problem of instability of superposed fluids. The remarkable point is its dynamical mechanism for the energy and momentum transport from the non-perturbed flow.

The relations of co-ordinate and notations of physical quantities are same as in chapter 2. In the present problem we set the non-perturbed uniform flow horizontally. In the upper fluid the velocity of this flow is $U^{(1)}$ and in the lower fluid it is $U^{(2)}$. We assume $U^{(1)}>U^{(2)}$. The perturbed motion may be considered irrotational except for the interface, where the velocity of the non-perturbed flow jumps. Therefore conditions of the perturbed motion (2-1), (2-2), (2-3) and (2-4) are also used in the present problem. Expressions of (2-5) and (2-6) are just modified by the existance of the non-perturbed flow $U^{(1)}$ and $U^{(2)}$.

In the present problem, at the interface,

$$\left.\begin{aligned}&\frac{1}{2}\rho^{(1)}U^{(1)2}+p_0^{(1)}=F^{(1)}(t)\\&\frac{1}{2}\rho^{(2)}U^{(2)2}+p_0^{(2)}=F^{(2)}(t)\end{aligned}\right\}\qquad(3\text{-}1)$$

Putting $p_0^{(1)}=p_0^{(2)}=0$, the dynamical condition at the interface is;

$$\rho^{(1)}\frac{\partial\phi^{(1)}}{\partial t}+\frac{1}{2}\rho^{(1)}u^{(1)2}+\frac{1}{2}\rho^{(1)}v^{(1)2}+\rho^{(1)}u^{(1)}U^{(1)}+\rho^{(1)}g\eta$$
$$=\rho^{(2)}\frac{\partial\phi^{(2)}}{\partial t}+\frac{1}{2}\rho^{(2)}u^{(2)2}+\frac{1}{2}\rho^{(2)}v^{(2)2}+\rho^{(2)}u^{(2)}U^{(2)}+\rho^{(2)}g\eta\quad\text{at}\quad y=\eta\qquad(3\text{-}2)$$

Kinematical conditions at the interface are;

$$\frac{\partial\eta}{\partial t}+(U^{(1)}+u^{(1)})\frac{\partial\eta}{\partial x}=v^{(1)}\qquad\text{at}\quad y=\eta\qquad(3\text{-}3)$$

$$\frac{\partial\eta}{\partial t}+(U^{(2)}+u^{(2)})\frac{\partial\eta}{\partial x}=v^{(2)}\qquad\text{at}\quad y=\eta\qquad(3\text{-}4)$$

From (3-2), (3-3) and (3-4),

$$\rho^{(1)}\frac{\partial\phi^{(1)}}{\partial t}+\frac{1}{2}\rho^{(1)}\left(\frac{\partial\phi^{(1)}}{\partial x}\right)^2+\frac{1}{2}\rho^{(1)}\left(\frac{\partial\phi^{(1)}}{\partial y}\right)^2+\rho^{(1)}\frac{\partial\phi^{(1)}}{\partial x}U^{(1)}+\rho^{(1)}g\eta$$
$$=\rho^{(2)}\frac{\partial\phi^{(2)}}{\partial t}+\frac{1}{2}\rho^{(2)}\left(\frac{\partial\phi^{(2)}}{\partial x}\right)^2+\frac{1}{2}\rho^{(2)}\left(\frac{\partial\phi^{(2)}}{\partial y}\right)^2+\rho^{(2)}\frac{\partial\phi^{(2)}}{\partial x}U^{(2)}+\rho^{(2)}g\eta\quad\text{at}\quad y=\eta$$
$$(3\text{-}5)$$

$$\frac{\partial\eta}{\partial t}+U^{(1)}\frac{\partial\eta}{\partial x}+\frac{\partial\phi^{(1)}}{\partial x}\frac{\partial\eta}{\partial x}=\frac{\partial\phi^{(1)}}{\partial y}\qquad\text{at}\quad y=\eta\qquad(3\text{-}6)$$

$$\frac{\partial\eta}{\partial t}+U^{(2)}\frac{\partial\eta}{\partial x}+\frac{\partial\phi^{(2)}}{\partial x}\frac{\partial\eta}{\partial x}=\frac{\partial\phi^{(2)}}{\partial y}\qquad \text{at}\quad y=\eta \tag{3-7}$$

A similar method of perturbation with (2-15) is used.

$$\left.\begin{aligned}\phi^{(1)}&=\alpha\phi_1{}^{(1)}+\alpha^2\phi_2{}^{(1)}+\alpha^3\phi_3{}^{(1)}+\cdots\cdots\\ \phi^{(2)}&=\alpha\phi_1{}^{(2)}+\alpha^2\phi_2{}^{(2)}+\alpha^3\phi_3{}^{(2)}+\cdots\cdots\\ \eta&=\alpha\eta_1+\alpha^2\eta_2+\alpha^3\eta_3+\cdots\cdots\end{aligned}\right\} \tag{3-8}$$

By making use of (3-8), the dynamical condition (3-5) is perturbed around $y=0$.

$$\begin{aligned}&\rho^{(1)}\frac{\partial\phi_1{}^{(1)}(0)}{\partial t}+\rho^{(1)}U^{(1)}\frac{\partial\phi_1{}^{(1)}(0)}{\partial x}+\rho^{(1)}g\eta_1\\ &=\rho^{(2)}\frac{\partial\phi_1{}^{(2)}(0)}{\partial t}+\rho^{(2)}U^{(2)}\frac{\partial\phi_1{}^{(2)}(0)}{\partial x}+\rho^{(2)}g\eta_1\end{aligned} \tag{3-9-1}$$

$$\begin{aligned}&\rho^{(1)}\frac{\partial\phi_2{}^{(1)}(0)}{\partial t}+\rho^{(1)}\frac{\partial^2\phi_1{}^{(1)}(0)}{\partial t\,\partial y}\eta_1+\frac{1}{2}\rho^{(1)}\left(\frac{\partial\phi_1{}^{(1)}(0)}{\partial x}\right)^2+\frac{1}{2}\rho^{(1)}\left(\frac{\partial\phi_1{}^{(1)}(0)}{\partial y}\right)^2\\ &+\rho^{(1)}U^{(1)}\frac{\partial\phi_2{}^{(1)}(0)}{\partial x}+\rho^{(1)}U^{(1)}\frac{\partial^2\phi_1{}^{(1)}(0)}{\partial x\,\partial y}\eta_1+\rho^{(1)}g\eta_2\\ &=\rho^{(2)}\frac{\partial\phi_2{}^{(2)}(0)}{\partial t}+\rho^{(2)}\frac{\partial^2\phi_1{}^{(2)}(0)}{\partial t\,\partial y}\eta_1+\frac{1}{2}\rho^{(2)}\left(\frac{\partial\phi_1{}^{(2)}(0)}{\partial x}\right)^2+\frac{1}{2}\rho^{(2)}\left(\frac{\partial\phi_1{}^{(2)}(0)}{\partial y}\right)^2\\ &+\rho^{(2)}U^{(2)}\frac{\partial\phi_2{}^{(2)}(0)}{\partial x}+\rho^{(2)}U^{(2)}\frac{\partial^2\phi_1{}^{(2)}(0)}{\partial x\,\partial y}\eta_1+\rho^{(2)}g\eta_2\end{aligned} \tag{3-9-2}$$

$$\begin{aligned}&\rho^{(1)}\frac{\partial\phi_3{}^{(1)}(0)}{\partial t}+\rho^{(1)}\frac{\partial^2\phi_2{}^{(1)}(0)}{\partial t\,\partial y}\eta_1+\rho^{(1)}\frac{\partial^2\phi_1{}^{(1)}(0)}{\partial t\,\partial y}\eta_2\\ &+\frac{1}{2}\rho^{(1)}\frac{\partial^3\phi_1{}^{(1)}(0)}{\partial t\,\partial y^2}\eta_1{}^2+\rho^{(1)}\frac{\partial\phi_1{}^{(1)}(0)}{\partial x}\frac{\partial\phi_2{}^{(1)}(0)}{\partial x}\\ &+\rho^{(1)}\frac{\partial\phi_1{}^{(1)}(0)}{\partial x}\frac{\partial^2\phi_1{}^{(1)}(0)}{\partial x\,\partial y}\eta_1+\rho^{(1)}\frac{\partial\phi_1{}^{(1)}(0)}{\partial y}\frac{\partial\phi_2{}^{(1)}(0)}{\partial y}\\ &+\rho^{(1)}\frac{\partial\phi_1{}^{(1)}(0)}{\partial y}\frac{\partial^2\phi_1{}^{(1)}(0)}{\partial y^2}\eta_1+\rho^{(1)}U^{(1)}\frac{\partial\phi_3{}^{(1)}(0)}{\partial x}\\ &+\rho^{(1)}U^{(1)}\frac{\partial^2\phi_2{}^{(1)}(0)}{\partial x\,\partial y}\eta_1+\rho^{(1)}U^{(1)}\frac{\partial^2\phi_1{}^{(1)}(0)}{\partial x\,\partial y}\eta_2+\frac{1}{2}\rho^{(1)}U^{(1)}\frac{\partial^3\phi_1{}^{(1)}(0)}{\partial x\,\partial y^2}\eta_1{}^2+\rho^{(1)}g\eta_3\\ &=\rho^{(2)}\frac{\partial\phi_3{}^{(2)}(0)}{\partial t}+\rho^{(2)}\frac{\partial^2\phi_2{}^{(2)}(0)}{\partial t\,\partial y}\eta_1+\rho^{(2)}\frac{\partial^2\phi_1{}^{(2)}(0)}{\partial t\,\partial y}\eta_2\\ &+\frac{1}{2}\rho^{(2)}\frac{\partial^3\phi_1{}^{(2)}(0)}{\partial t\,\partial y^2}\eta_1{}^2+\rho^{(2)}\frac{\partial\phi_1{}^{(2)}(0)}{\partial x}\frac{\partial\phi_2{}^{(2)}(0)}{\partial x}\\ &+\rho^{(2)}\frac{\partial\phi_1{}^{(2)}(0)}{\partial x}\frac{\partial^2\phi_1{}^{(2)}(0)}{\partial x\,\partial y}\eta_1+\rho^{(2)}\frac{\partial\phi_1{}^{(2)}(0)}{\partial y}\frac{\partial\phi_2{}^{(2)}(0)}{\partial y}\\ &+\rho^{(2)}\frac{\partial\phi_1{}^{(2)}(0)}{\partial y}\frac{\partial^2\phi_1{}^{(2)}(0)}{\partial y^2}\eta_1+\rho^{(2)}U^{(2)}\frac{\partial\phi_3{}^{(2)}(0)}{\partial x}\end{aligned}$$

$$+\rho^{(2)}U^{(2)}\frac{\partial^2\phi_2^{(2)}(0)}{\partial x\,\partial y}\eta_1+\rho^{(2)}U^{(2)}\frac{\partial^2\phi_1^{(2)}(0)}{\partial x\,\partial y}\eta_2+\frac{1}{2}\rho^{(2)}U^{(2)}\frac{\partial^3\phi_1^{(2)}(0)}{\partial x\,\partial y^2}\eta_1^2+\rho^{(2)}g\eta_3 \tag{3-9-3}$$

From (3-6) and (3-7) similarly,

$$\frac{\partial\eta_1}{\partial t}+U^{(1)}\frac{\partial\eta_1}{\partial x}=\frac{\partial\phi_1^{(1)}(0)}{\partial y} \tag{3-10-1}$$

$$\frac{\partial\eta_2}{\partial t}+U^{(1)}\frac{\partial\eta_2}{\partial x}+\frac{\partial\phi_1^{(1)}(0)}{\partial x}\frac{\partial\eta_1}{\partial x}=\frac{\partial\phi_2^{(1)}(0)}{\partial y}+\frac{\partial^2\phi_1^{(1)}(0)}{\partial y^2}\eta_1 \tag{3-10-2}$$

$$\frac{\partial\eta_3}{\partial t}+U^{(1)}\frac{\partial\eta_3}{\partial x}+\frac{\partial\phi_2^{(1)}(0)}{\partial x}\frac{\partial\eta_1}{\partial x}+\frac{\partial^2\phi_1^{(1)}(0)}{\partial x\,\partial y}\eta_1\frac{\partial\eta_1}{\partial x}+\frac{\partial\phi_1^{(1)}(0)}{\partial x}\frac{\partial\eta_2}{\partial x}$$
$$=\frac{\partial\phi_3^{(1)}(0)}{\partial y}+\frac{\partial^2\phi_2^{(1)}(0)}{\partial y^2}\eta_1+\frac{\partial^2\phi_1^{(1)}(0)}{\partial y^2}\eta_2+\frac{1}{2}\frac{\partial^3\phi_1^{(1)}(0)}{\partial y^3}\eta_1^2 \tag{3-10-3}$$

$$\frac{\partial\eta_1}{\partial t}+U^{(2)}\frac{\partial\eta_1}{\partial x}=\frac{\partial\phi_1^{(2)}(0)}{\partial y} \tag{3-11-1}$$

$$\frac{\partial\eta_2}{\partial t}+U^{(2)}\frac{\partial\eta_2}{\partial x}+\frac{\partial\phi_1^{(2)}(0)}{\partial x}\frac{\partial\eta_1}{\partial x}=\frac{\partial\phi_2^{(2)}(0)}{\partial y}+\frac{\partial^2\phi_1^{(2)}(0)}{\partial y^2}\eta_1 \tag{3-11-2}$$

$$\frac{\partial\eta_3}{\partial t}+U^{(2)}\frac{\partial\eta_3}{\partial x}+\frac{\partial\phi_2^{(2)}(0)}{\partial x}\frac{\partial\eta_1}{\partial x}+\frac{\partial^2\phi_1^{(2)}(0)}{\partial x\,\partial y}\eta_1\frac{\partial\eta_1}{\partial x}+\frac{\partial\phi_1^{(2)}(0)}{\partial x}\frac{\partial\eta_2}{\partial x}$$
$$=\frac{\partial\phi_3^{(2)}(0)}{\partial y}+\frac{\partial^2\phi_2^{(2)}(0)}{\partial y^2}\eta_1+\frac{\partial^2\phi_1^{(2)}(0)}{\partial y^2}\eta_2+\frac{1}{2}\frac{\partial^3\phi_1^{(2)}(0)}{\partial y^3}\eta_1^2 \tag{3-11-3}$$

The linear relations (the approximations of the first order) can be obtained by making use of (3-9-1), (3-10-1) and (3-11-1).

Putting,

$$\left.\begin{aligned}&\phi_1^{(1)}=\boldsymbol{R}\{B_1^{(1)}e^{ik(x-ct)}\cdot e^{-ky}\}\qquad (k>0,\ c=c_r+ic_i)\\&\phi_1^{(2)}=\boldsymbol{R}\{B_1^{(2)}e^{ik(x-ct)}\cdot e^{ky}\}\\&\quad\eta=\boldsymbol{R}\{A_1e^{ik(x-ct)}\}\end{aligned}\right\} \tag{3-12}$$

A_1 : real and positive

From (3-10-1) and (3-11-1)

$$\left.\begin{aligned}&\phi_1^{(1)}=\boldsymbol{R}\{-iA_1(U^{(1)}-c)\,e^{ik(x-ct)}\cdot e^{-ky}\}\\&\phi_1^{(2)}=\boldsymbol{R}\{iA_1(U^{(2)}-c)\,e^{ik(x-ct)}\cdot e^{ky}\}\end{aligned}\right\} \tag{3-13}$$

c can be determined by (3-9-1);

$$c=\frac{\rho^{(1)}U^{(1)}+\rho^{(2)}U^{(2)}}{\rho^{(1)}+\rho^{(2)}}\pm\sqrt{\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}\frac{g}{k}-\frac{\rho^{(1)}\rho^{(2)}(U^{(1)}-U^{(2)})^2}{(\rho^{(1)}+\rho^{(2)})^2}} \tag{3-14}$$

In the present problem the second term of (3-14) should be imaginary, and stillmore we only treat the case of $c_i>0$. Therefore

$$c=c_r+ic_i=\frac{\rho^{(1)}U^{(1)}+\rho^{(2)}U^{(2)}}{\rho^{(1)}+\rho^{(2)}}+i\left\{\frac{\rho^{(1)}\rho^{(2)}(U^{(1)}-U^{(2)})^2}{(\rho^{(2)}+\rho^{(1)})^2}-\frac{g}{k}\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}\right\}^{1/2} \tag{3-15}$$

Here $\frac{\rho^{(1)}\rho^{(2)}(U^{(1)}-U^{(2)})^2}{(\rho^{(1)}+\rho^{(2)})^2}-\frac{g}{k}\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}>0$ is the condition for the existence of instability.

(3-12) may be rewritten to

$$\left.\begin{aligned}\phi_1{}^{(1)}&=A_1(U^{(1)}-c_r)\sin k(x-c_rt)\,e^{kc_it}\cdot e^{-ky}-A_1c_i\cos k(x-c_rt)\,e^{kc_it}\cdot e^{-ky}\\ \phi_1{}^{(2)}&=-A_1(U^{(2)}-c_r)\sin k(x-c_rt)\,e^{kc_it}\cdot e^{ky}+A_1c_i\cos k(x-c_rt)\,e^{kc_it}\,e^{ky}\\ \eta_1&=A_1\cos k(x-c_rt)\,e^{kc_it}\end{aligned}\right\}\quad(3\text{-}16)$$

3.2 The dynamical processes of wave amplification of the Kelvin-Helmholtz instability

The perturbed flow of the present problem is amplified by the rate of kc_i per unit time. Here kc_i is expressed by (3-15). The expression of (3-15) indicates that c_i increases with the increase of $|U^{(1)}-U^{(2)}|$, and at this point the mechanism of instability of the present type is not so different from the instability of general shear flow of heterogeneous fluid. The perturbed flow must be amplified by the transfer of mechanical energy from the non-perturbed flow.

In the Kelvin-Helmholtz instability the value of c_i/c_r is not always small, and c_i may be able to grow up extensively even in the linear treatment. Therefore we cannot put $(c_i/c_r)^2\rightarrow 0$ in the present problem. On the contrary the amplification mechanism of the Tollmien-Schlichting wave is treated at the condition $(c_i/c_r)^2\rightarrow 0$ in its linear treatment.

In concern with the Kelvin-Helmholtz instability, J. W. Miles (1959) made an attempt to extend it to the case of more general velocity profile of the horizontal flow. The dynamical foundation of his treatment may be obtained at the condition of very samll c_i (almost zero). The present analysis proceeds along the different way by making use of the uniform volocity profile, and the comprehensive interpretation is found including the general case of c_i/c_r.

The kinetic energy of the perturbed wave motion in the present problem may be computed as

$$\frac{1}{4}(\rho^{(2)}-\rho^{(1)})gA_1{}^2e^{2kc_it}+\frac{1}{2}(\rho^{(1)}+\rho^{(2)})A_1{}^2kc_i{}^2e^{2kc_it}\quad(3\text{-}17)$$

The potential energy is

$$\frac{1}{2}(\rho^{(2)}-\rho^{(1)})gA_1{}^2\cos^2 k(x-c_rt)e^{2kc_it}\quad(3\text{-}18)$$

and so the time average of the rate of the increase of the mechanical energy $\left(\overline{\frac{dE}{dt}}\right)$ can be expressed by

$$\overline{\frac{dE}{dt}}=(\rho^{(2)}-\rho^{(1)})gA_1{}^2kc_ie^{2kc_it}+(\rho^{(1)}+\rho^{(2)})A_1{}^2k^2c_i{}^3e^{2kc_it}\quad(3\text{-}19)$$

Then we compute the momentum transport of the perturbed wave motion. This may be expressed at the upper fluid,

$$\left.\begin{aligned} M^{(1)}&=-\frac{1}{2}A_1^2k\rho^{(1)}\rho^{(2)}\frac{U^{(1)}-U^{(2)}}{\rho^{(1)}+\rho^{(2)}}e^{2kc_it} \\ \frac{dM^{(1)}}{dt}&=-A_1^2k^2c_i\rho^{(1)}\rho^{(2)}\frac{U^{(1)}-U^{(2)}}{\rho^{(1)}+\rho^{(2)}}e^{2kc_it} \end{aligned}\right\} \tag{3-20}$$

At the lower fluid,

$$\left.\begin{aligned} M^{(2)}&=\frac{1}{2}A_1^2k\rho^{(1)}\rho^{(2)}\frac{U^{(1)}-U^{(2)}}{\rho^{(1)}+\rho^{(2)}}e^{2kc_it} \\ \frac{dM^{(2)}}{dt}&=A_1^2k^2c_i\rho^{(1)}\rho^{(2)}\frac{U^{(1)}-U^{(2)}}{\rho^{(1)}+\rho^{(2)}}e^{2kc_it} \end{aligned}\right\} \tag{3-21}$$

The momentum transport expressed by (3-20) and (3-21) is one of the special characteristics of growing perturbed waves in the Kelvin-Helmholts instability. The momentum transport of the upper layer is negative at the condition of $U^{(1)} > U^{(2)}$, and its absolute value is equal to that of the lower fluid. Stillmore $M^{(2)}$ of the lower fluid becomes zero when $\rho^{(1)}$ of the upper fluid becomes negligible.

The momentum transport of the interfacial waves, which is not amplified by the Kelvin-Helmholtz instability, is quite different from the above two expressions. A good example is the case of $U^{(1)}=U^{(2)}=0$, and so c_i is equal to zero. In this case the momentum transport of the upper fluid is ;

$$M^{(1)}=\frac{1}{2}A_1^2k\rho^{(1)}\sqrt{\frac{g}{k}\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(1)}+\rho^{(2)}}} \tag{3-22}$$

The momentum transport of the lower fluid is ;

$$M^{(2)}=\frac{1}{2}A_1^2k\rho^{(2)}\sqrt{\frac{g}{k}\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(1)}+\rho^{(2)}}} \tag{3-23}$$

By making use of the expression $E=\frac{1}{2}(\rho^{(2)}-\rho^{(1)})gA_1^2$, $c=\sqrt{\frac{g}{k}\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(1)}+\rho^{(2)}}}$, (3-22) and (3-23) may have a well known simple relation $E=c(M^{(1)}+M^{(2)})=cM$, and this is same as the case of surface waves. But at the condition of the Kelvin-Helmholtz instability $M^{(1)}+M^{(2)}=0$ is consistent.

It may be considered that the transfer of mechanical energy from the non-perturbed flow to the perturbed wave motion is made by the action of Reynolds stress at the present inviscid condition. In the interior of both fluids this stress is not active even at the present unstable condition, because we have assumed that both fluids make an irrotational motion. At the interface this stress may be controlled by $\overline{-(u_1^{(1)}-u_1^{(2)})(v_1^{(1)}-v_1^{(2)})}$ at $y=0$.

From (3-16),

$$\begin{aligned} &\overline{-(u_1^{(1)}-u_1^{(2)})(v_1^{(1)}-v_1^{(2)})} \text{ at } y=0 \\ &=-\{-A_1^2(U^{(1)}-c_r)c_ik^2e^{2kc_it}+A_1^2(U^{(2)}-c_r)c_ik^2e^{2kc_it}\} \end{aligned} \tag{3-24}$$

When we compare the two terms of the right hand side of (3-24) with (3-20) and (3-21), we can understand that the first term of (3-24) acts on the upper surface of interfacial boundary (it is influential to the momentum change of the lower fluid), and that the second term acts on the lower surface of interfacial boundary (its effect is active to the momentum change of the upper fluid). In other words the first term of (3-24) multiplied by $\rho^{(1)}$ is equal to (3-21), and the second term of (3-24), multiplied by $\rho^{(2)}$ and if its sign is changed, is in agreement with (3-20).

By this way the Reynolds stress which acts to the direction of progress of waves on the upper surface of interfacial boundary is

$$\tau_{(+0)}=\rho^{(1)}A_1^2(U^{(1)}-c_r)c_ik^2e^{2kc_it} \tag{3-25}$$

and the one which works to the inverse direction of progress of waves along the lower surface of interfacial boundary is

$$\tau_{(-0)}=-\rho^{(2)}A_1^2(U^{(2)}-c_r)c_ik^2e^{2kc_it} \tag{3-26}$$

The work per unit time by these stresses is computed as

$$\tau_{(+0)}\cdot U^{(1)}-\tau_{(-0)}\cdot U^{(2)}=(\rho^{(2)}-\rho^{(1)})gA_1^2c_ike^{2kc_it}+(\rho^{(1)}+\rho^{(2)})A_1^2c_i^3k^2e^{2kc_it} \tag{3-27}$$

(3-27) is in agreement with (3-19).

The work at the flexible boundary, which is caused by the Reynolds stress in the present case, must be appreciated as the work of pressure fluctuation at the same boundary.

In the present problem at the interface,

$$\left.\begin{aligned} p_1^{(1)}&=-\rho^{(1)}\frac{\partial\phi_1^{(1)}}{\partial t}-\rho^{(1)}u^{(1)}U^{(1)}-\rho^{(1)}g\eta_1 \\ p_1^{(2)}&=-\rho^{(2)}\frac{\partial\phi_1^{(2)}}{\partial t}-\rho^{(2)}u^{(2)}U^{(2)}-\rho^{(2)}g\eta_1 \end{aligned}\right\} \tag{3-28}$$

These are computed as ;

$$\begin{aligned} p_1^{(1)}=&-\rho^{(1)}A_1kc_i(U^{(1)}-c_r)\sin k(x-c_rt)e^{kc_it} \\ &+\rho^{(1)}A_1(U^{(1)}-c_r)kc_r\cos k(x-c_rt)e^{kc_it} \\ &+\rho^{(1)}A_1kc_i^2\cos k(x-c_rt)e^{kc_it}+\rho^{(1)}A_1c_ikc_r\sin k(x-c_rt)e^{kc_it} \\ &-\rho^{(1)}U^{(1)}A_1k(U^{(1)}-c_r)\cos k(x-c_rt)e^{kc_it} \\ &-\rho^{(1)}U^{(1)}A_1kc_i\sin k(x-c_rt)e^{kc_it}-\rho^{(1)}gA_1\cos k(x-c_rt)e^{kc_it} \end{aligned} \tag{3-29}$$

$$\begin{aligned} p_1^{(2)}=&\rho^{(2)}A_1kc_i(U^{(2)}-c_r)\sin k(x-c_rt)e^{kc_it} \\ &-\rho^{(2)}A_1(U^{(2)}-c_r)kc_r\cos k(x-c_rt)e^{kc_it} \\ &-\rho^{(2)}A_1kc_i^2\cos k(x-c_rt)e^{kc_it}-\rho^{(2)}A_1c_ikc_r\sin k(x-c_rt)e^{kc_it} \\ &+\rho^{(2)}U^{(2)}A_1k(U^{(2)}-c_r)\cos k(x-c_rt)e^{kc_it} \\ &+\rho^{(2)}U^{(2)}A_1kc_i\sin k(x-c_rt)e^{kc_it}-\rho^{(2)}gA_1\cos k(x-c_rt)e^{kc_it} \end{aligned} \tag{3-30}$$

By making use of these expressions and vertical velocities from (3-16),

$$-\overline{p_1^{(1)}v_1^{(2)}}=\rho^{(1)}A_1^2k^2c_i^3e^{2kc_it}+\rho^{(1)}A_1^2kc_ig\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(1)}+\rho^{(2)}}e^{2kc_it}$$

$$-\frac{1}{2}\rho^{(1)}A_1^2k^2c_i^3e^{2kc_it}+\frac{1}{2}\rho^{(1)}gA_1^2kc_ie^{2kc_it}$$

$$+\frac{1}{2}\rho^{(2)}A_1^2k^2c_i^3e^{2kc_it}+\frac{1}{2}\rho^{(2)}gA_1^2kc_i\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(1)}+\rho^{(2)}}e^{2kc_it} \tag{3-31}$$

$$\overline{p_1^{(2)}v_1^{(1)}}=\rho^{(2)}A_1^2k^2c_i^3e^{2kc_it}+\rho^{(2)}A_1^2kc_ig\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(1)}+\rho^{(2)}}e^{2kc_it}$$

$$-\frac{1}{2}\rho^{(2)}A_1^2k^2c_i^3e^{2kc_it}-\frac{1}{2}\rho^{(2)}gA_1^2kc_ie^{2kc_it}$$

$$+\frac{1}{2}\rho^{(1)}A_1^2k^2c_i^3e^{2kc_it}+\frac{1}{2}\rho^{(1)}A_1^2kc_ig\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(1)}+\rho^{(2)}}e^{2kc_it} \tag{3-32}$$

In (3-31) and (3-32) the first and the second terms of the righthand side are due to components of the same phase with the slope of wave profile, and the other terms are from components with the phase of wave profile. From (3-31) and (3-32) the work per unit time at the interfacial boundary given by the pressure fluctuation is

$$-\overline{p_1^{(1)}v_1^{(2)}}+\overline{p_1^{(2)}v_1^{(1)}}=(\rho^{(1)}+\rho^{(2)})A_1^2k^2c_i^3e^{2kc_it}+A_1^2kc_ig(\rho^{(2)}-\rho^{(1)})e^{2kc_it} \tag{3-33}$$

This result coincides with (3-19) and (3-27), and it is clear from (3-31) and (3-32) that the effective work of (3-33) is given by components of the same phase with the slope of wave profile. This is a noticeable point and is different from some previous discussion, which is effective at the stage of very small c_i (see J. W. Miles (1959) pp. 584~585).

The result of these computations indicates that the Kelvin-Helmholtz instability may be an amplification mechanism which satisfies the necessitated dynamical conditions of momentum and energy transfer in its linear approximation. But the assumption for the calculation of this instability is very strict, and in actual phenomena this strictness and the peculiarity of the assumption may limit the practical application severely.

3.3 The second order approximation and its remarkable properties

(3.3-1) Putting the first order approximations ;

$$\phi_1^{(1)}=\left\{\frac{-iA_1}{2}(U^{(1)}-c_r)-\frac{A_1}{2}c_i\right\}e^{ik(x-c_rt)}\cdot e^{kc_it}\cdot e^{-ky}$$

$$+\left\{\frac{iA_1}{2}(U^{(1)}-c_r)-\frac{A_1}{2}c_i\right\}e^{-ik(x-c_rt)}\cdot e^{kc_it}\cdot e^{-ky} \tag{3-34}$$

$$\phi_1^{(2)}=\left\{\frac{iA_1}{2}(U^{(2)}-c_r)+\frac{A_1}{2}c_i\right\}e^{ik(x-c_rt)}\cdot e^{kc_it}\cdot e^{ky}$$

$$+\left\{\frac{-iA_1}{2}(U^{(2)}-c_r)+\frac{A_1}{2}c_i\right\}e^{-ik(x-c_rt)}\cdot e^{kc_it}\cdot e^{ky} \tag{3-35}$$

$$\eta_1=\frac{A_1}{2}e^{ik(x-c_rt)}\cdot e^{kc_it}+\frac{A_1}{2}e^{-ik(x-c_rt)}\cdot e^{kc_it} \tag{3-36}$$

The second order approximation may be also expressed by ;

$$\phi_2^{(1)}=(B_{21}^{(1)}+iB_{22}^{(1)})e^{i2k(x-c_rt)}\cdot e^{2kc_it}\cdot e^{-2ky}$$

$$+(B_{21}^{(1)}-iB_{22}^{(1)})e^{-i2k(x-c_rt)}\cdot e^{2kc_it}\cdot e^{-2ky}+\text{Const}_2^{(1)}t \qquad (3\text{-}37)$$

$$\phi_2^{(2)}=(B_{21}^{(2)}+iB_{22}^{(2)})e^{i2k(x-c_rt)}\cdot e^{2kc_it}\cdot e^{2ky}$$

$$+(B_{21}^{(2)}-iB_{22}^{(2)})e^{-i2k(x-c_rt)}\cdot e^{2kc_it}\cdot e^{2ky}+\text{Const}_2^{(2)}t \qquad (3\text{-}38)$$

$$\eta_2=(A_{21}+iA_{22})e^{i2k(x-c_rt)}\cdot e^{2kc_it}+(A_{21}-iA_{22})e^{-i2k(x-c_rt)}\cdot e^{2kc_it} \qquad (3\text{-}39)$$

Using the second order kinematic conditions of interfacial boundary (3-10-2) and (3-11-2),

$$\left.\begin{aligned} B_{21}^{(1)}&=A_{22}(U^{(1)}-c_r)-A_{21}c_i-\frac{1}{4}kA_1^2c_i \\ B_{22}^{(1)}&=A_{21}(c_r-U^{(1)})-A_{22}c_i-\frac{1}{4}kA_1^2(U^{(1)}-c_r) \end{aligned}\right\} \qquad (3\text{-}40)$$

$$\left.\begin{aligned} B_{21}^{(2)}&=A_{22}(c_r-U^{(2)})+A_{21}c_i-\frac{1}{4}kA_1^2c_i \\ B_{22}^{(2)}&=A_{21}(U^{(2)}-c_r)+A_{22}c_i-\frac{1}{4}kA_1^2(U^{(2)}-c_r) \end{aligned}\right\} \qquad (3\text{-}41)$$

Using (3-9-2), next three relations can be obtained ;

$$\rho^{(1)}2kc_rB_{22}^{(1)}+\rho^{(1)}2kc_iB_{21}^{(1)}+\rho^{(1)}k^2\frac{A_1^2}{4}(U^{(1)}-c_r)c_r+\rho^{(1)}k^2c_i^2\frac{A_1^2}{4}$$

$$-\rho^{(1)}U^{(1)}2kB_{22}^{(1)}-\rho^{(1)}U^{(1)}k^2\frac{A_1^2}{4}(U^{(1)}-c_r)+\rho^{(1)}gA_{21}$$

$$=\rho^{(2)}2kc_rB_{22}^{(2)}+\rho^{(2)}2kc_iB_{21}^{(2)}+\rho^{(2)}k^2\frac{A_1^2}{4}(U^{(2)}-c_r)c_r+\rho^{(2)}k^2c_i^2\frac{A_1^2}{4}$$

$$-\rho^{(2)}U^{(2)}2kB_{22}^{(2)}-\rho^{(2)}U^{(2)}k^2\frac{A_1^2}{4}(U^{(2)}-c_r)+\rho^{(2)}gA_{21} \qquad (3\text{-}42)$$

$$\rho^{(1)}2kc_iB_{22}^{(1)}-\rho^{(1)}2kc_rB_{21}^{(1)}-\rho^{(1)}k^2c_rc_i\frac{A_1^2}{4}+\rho^{(1)}k^2\frac{A_1^2}{4}(U^{(1)}-c_r)c_i$$

$$+\rho^{(1)}U^{(1)}2kB_{21}^{(1)}+\rho^{(1)}U^{(1)}k^2\frac{A_1^2}{4}c_i+\rho^{(1)}gA_{22}$$

$$=\rho^{(2)}2kc_iB_{22}^{(2)}-\rho^{(2)}2kc_rB_{21}^{(2)}+\rho^{(2)}k^2c_i\frac{A_1^2}{4}(U^{(2)}-c_r)-\rho^{(2)}k^2c_rc_i\frac{A_1^2}{4}$$

$$+\rho^{(2)}U^{(2)}2kB_{21}^{(2)}+\rho^{(2)}U^{(2)}k^2\frac{A_1^2}{4}c_i+\rho^{(2)}gA_{22} \qquad (3\text{-}43)$$

$$\rho^{(1)}\,\text{Const}_2^{(1)}+\rho^{(1)}k^2A_1^2c_i^2=\rho^{(2)}\,\text{Const}_2^{(2)}+\rho^{(2)}k^2A_1^2c_i^2 \qquad (3\text{-}44)$$

From (3-44)

$$\text{Const}_2^{(1)}=\text{Const}_2^{(2)}=-k^2A_1^2c_i^2 \qquad (3\text{-}45)$$

By this way, if $c_i \neq 0$ is consistent, $\mathrm{Const}_2^{(1)}$ and $\mathrm{Const}_2^{(2)}$ are proportional to c_i^2 even in a condition that the both layers are sufficiently deep. Inserting relations (3-40) and (3-41) into (3-42) and (3-43), A_{22} and A_{21} may be determined as

$$\left.\begin{aligned} A_{22} &= \frac{c_i\left(\dfrac{AD}{E^2}+\dfrac{B}{E}\right)}{1+\dfrac{A^2c_i^2}{E^2}} \\ A_{21} &= \frac{\dfrac{ABc_i^2}{E^2}-\dfrac{D}{E}}{1+\dfrac{A^2c_i^2}{E^2}} \end{aligned}\right\} \tag{3-46}$$

Here

$$\left.\begin{aligned} &4k\{\rho^{(1)}(U^{(1)}-c_r)+\rho^{(2)}(U^{(2)}-c_r)\}=A \\ &\frac{k^2A_1^2}{2}\{\rho^{(2)}(U^{(2)}-c_r)-\rho^{(1)}(U^{(1)}-c_r)\}=B \\ &\frac{k^2A_1^2}{4}\{\rho^{(2)}((U^{(2)}-c_r)^2-c_i^2)-\rho^{(1)}((U^{(1)}-c_r)^2-c_i^2)\}=D \\ &g(\rho^{(2)}-\rho^{(1)})-2k\{\rho^{(2)}(U^{(2)}-c_r)^2+\rho^{(1)}(U^{(1)}-c_r)^2\}+2kc_i^2(\rho^{(2)}+\rho^{(1)})=E \end{aligned}\right\} \tag{3-47}$$

In the deduction of (3-46) of the second order approximation, we used the relation of $c=c_r+ic_i$, and c_i is treated as a finite one. But the concrete relation of the first order approximation (3-15) is not yet used. Therefore we can also use (3-46) at the condition of $U^{(1)}=U^{(2)}=0$, $c_i=0$. In this case c_r may be put $c_r^2=\dfrac{g}{k}\dfrac{\rho^{(2)}-\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}$, and A_{22} and A_{21} may be expressed by

$$\left.\begin{aligned} A_{22} &= 0 \\ A_{21} &= \frac{kA_1^2}{4}\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}} \end{aligned}\right\} \tag{3-48}$$

The expression of (3-48) is in agreement with (2-23), and the term with a phase shifted $\pi/2$ is disappeared. The appearance of A_{22} in the case of $c_i \neq 0$ should be remarked.

Then we use the relation (3-15) to obtain the clear meaning of (3-46) and (3-47) in the case of the Kelvin-Helmholtz instability.

Here

$$c_r=\frac{\rho^{(1)}U^{(1)}+\rho^{(2)}U^{(2)}}{\rho^{(1)}+\rho^{(2)}}, \quad c_i^2=\frac{\rho^{(1)}\rho^{(2)}(U^{(1)}-U^{(2)})^2}{(\rho^{(2)}+\rho^{(1)})^2}-\frac{g}{k}\frac{\rho^{(2)}-\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}} \quad (>0) \tag{3-49}$$

From (3-47) and (3-49),

$$A=0 \tag{3-50}$$

Therefore (3-46) may be simplified as

$$A_{22}=c_i\frac{B}{E},\qquad A_{21}=-\frac{D}{E} \tag{3-51}$$

(3.3-2) Here the properties of A_{22} and A_{21} of (3-51) will be examined at the condition that $(c_i/c_r)^2$ may be considered zero (c_i is very small).

From (3-47) and (3-49),

$$E=\frac{k\rho^{(1)}\rho^{(2)}(U^{(1)}-U^{(2)})^2}{\rho^{(2)}+\rho^{(1)}}-2k\{\rho^{(2)}(U^{(2)}-c_r)^2+\rho^{(1)}(U^{(1)}-c_r)^2\}$$

$$=\frac{-2k\rho^{(2)2}(U^{(2)}-c_r)^2-2k\rho^{(1)2}(U^{(1)}-c_r)^2-k\rho^{(1)}\rho^{(2)}\{(U^{(1)}+U^{(2)})-2c_r\}^2}{\rho^{(2)}+\rho^{(1)}}$$

$$<0 \tag{3-52}$$

(3-52) indicates that E is always negative in the present case.

Similarly,

$$D=\frac{k^2A_1^2}{4}\frac{(U^{(1)}-U^{(2)})^2\rho^{(1)}\rho^{(2)}(\rho^{(1)}-\rho^{(2)})}{(\rho^{(1)}+\rho^{(2)})^2}<0 \tag{3-53}$$

Accordingly $A_{21}<0$ is eatablished in the present case. On the contrary if $U^{(1)}=U^{(2)}=0$, and $c_i=0$ are consistent, A_{21} is positive as shown by (3-48). By this way under the condition of the Kelvin-Helmholtz instability, a quite peculiar wave form may develop, if way consider its second order approximation.

The expression of B of (3-47) is computed as

$$B=k^2A_1^2\frac{\rho^{(1)}\rho^{(2)}(U^{(2)}-U^{(1)})}{\rho^{(1)}+\rho^{(2)}} \tag{3-54}$$

Thus B changes its sign according to the sign of $(U^{(2)}-U^{(1)})$, and as we assume $U^{(1)}>U^{(2)}$, $B<0$ is consistent. $c_i>0$ is clear in the present problem, and so we have $A_{22}>0$. This is a noticeable result.

From (3-39) the positive sign of A_{22} indicates that the upper part of wave form deforms to windward side in inverse to its direction of progress at the second order approximation.

Expressions of A_{21} and A_{22} are

$$A_{21}=-\frac{D}{E}=\frac{-\frac{kA_1^2}{4}g(\rho^{(2)}-\rho^{(1)})^2}{2k\rho^{(2)2}(U^{(2)}-c_r)^2+2k\rho^{(1)2}(U^{(1)}-c_r)^2+k\rho^{(1)}\rho^{(2)}\{(U^{(1)}+U^{(2)})-2c_r\}^2} \tag{3-55}$$

$$A_{22}=c_i\frac{B}{E}=\frac{kA_1^2\rho^{(1)}\rho^{(2)}(U^{(1)}-U^{(2)})c_i}{2\rho^{(2)2}(U^{(2)}-c_r)^2+2\rho^{(1)2}(U^{(1)}-c_r)^2+\rho^{(1)}\rho^{(2)}\{(U^{(1)}+U^{(2)})-2c_r\}^2} \tag{3-56}$$

Under the condition of $(c_i/c_r)^2\to 0$, if $\rho^{(1)}$ approaches to $\rho^{(2)}$, these relations become

$$A_{21}\to 0\,,\quad A_{22}\to\frac{kA_1^2}{U^{(1)}-U^{(2)}}c_i\,,\quad c_r\to\frac{U^{(1)}+U^{(2)}}{2} \tag{3-57}$$

When the interfacial wave generated by the Kelvin-Helmholtz instability is seen at the boundary between fresh and salt water, the wave profile of the second order approximation may be controlled by (3-57).

At the condition of $\rho^{(1)}\to 0$ and of $\rho^{(2)}\to 1$ in (3-55) and (3-56), we have

$$A_{21}\to \text{order of } \left(-\frac{k}{4}A_1^2\right),\quad A_{22}\to 0,\quad c_r\to U^{(2)} \tag{3-58}$$

When the wave generated by the Kelvin-Helmholtz instability is seen at the boundary between air and water, its wave profile may be characterised by (3-58). It shows that, although the deformation of wave profile caused by A_{22} is not vivid, A_{21} is still active, and that the water wave is flat at its crest and steep at its trough. This effect of A_{21} of (3-58) is contrary to the usual appearance of wind generated waves of water surface.

The Kelvin-Helmholtz instability has generally very special characters in its first order approximation as given in section 3.1 and 3.2. The most distinguished one is that the real wave celerity (c_r) is not dispersive and does not depend on gravity in its expression. In the second order approximation we have also above-mentioned singular properties. S. Chandrasekhar (1961) introduced the picture (Fig. 117 of his book) taken by J. R. D. Francis as an example of the Kelvin-Helmholtz instability on oil surface. Although the capillary effect may be important at such initial stage of wave generation, it seems that the wave profile is not agreeable with the computed result of (3-58).

(3.3-3) In (3.3-2) we used the assumption of $(c_i/c_r)^2\to 0$, and so from (3-49) it is clear that, if $\rho^{(1)}$ approaches to $\rho^{(2)}$ beyond the degree of the case of fresh and salt water, $(U^{(1)}-U^{(2)})^2\to 0$ must be consistent. In such case $U^{(1)}-U^{(2)}$ becomes also sufficiently small. This means the rapid increase of A_{22} in (3-57). At the same time under the condition of $(c_i/c_r)^2\to 0$, the case of $\dfrac{g}{k}\dfrac{\rho^{(2)}-\rho^{(1)}}{\rho^{(2)}+\rho^{(1)}}>c_i^2$ should be considered in (3-49), and so $\rho^{(1)}$ cannot approach to $\rho^{(2)}$ indefinitely. Therefore we must desert the assumption of $(c_i/c_r)^2\to 0$, when we consider $\rho^{(1)}$ to approch to $\rho^{(2)}$ extremely.

Then we examine the behaviour of A_{21} and A_{22} when $\rho^{(1)}$ approaches to $\rho^{(2)}$ indefinitely. At the limit of this case, from (3-49)

$$c_r=\frac{U^{(1)}+U^{(2)}}{2},\quad c_i^2=\frac{(U^{(1)}-U^{(2)})^2}{4},\quad c_i=\frac{U^{(1)}-U^{(2)}}{2}\quad (c_i>0) \tag{3-59}$$

and, from (3-47)

$$\left.\begin{aligned} &A=0\\ &B=\frac{k^2A_1^2}{2}\rho^{(1)}(U^{(2)}-U^{(1)})\\ &D=0\\ &E=0 \end{aligned}\right\} \tag{3-60}$$

We cannot clarify the properties of A_{21} and A_{22} of (3-51) by making use of the result of (3-60). In (3-60) A is always zero at the condition of the Kelvin-Helmholtz instability, and so it does not need the more strict analysis. The expression of B in (3-60) does not change when $\rho^{(1)}$ approaches to $\rho^{(2)}$, and B has a negative finite value at the present assumption of $U^{(1)}>U^{(2)}$. D and E need the more detailed analysis, and must be examined at the limit of $(\rho^{(2)}-\rho^{(1)})\to 0$.

Putting

$$c_r=c_{r0}+\Delta c_r\,,\qquad c_i=c_{i0}+\Delta c_i \quad \text{and} \quad \rho^{(2)}-\rho^{(1)}=\Delta\rho \quad (\Delta\rho \text{ is very small})$$

$$\left(\text{here}\quad c_{r0}=\frac{U^{(1)}+U^{(2)}}{2}\,,\qquad c_{i0}=\frac{U^{(1)}-U^{(2)}}{2}\qquad\qquad \text{from (3-59)}\right)$$

the following expressions of Δc_r, Δc_i can be obtained from (3-49).

$$\Delta c_r=\frac{\Delta\rho}{\rho^{(1)}}\frac{U^{(2)}-U^{(1)}}{4} \tag{3-61}$$

$$\Delta c_i=-\frac{g}{k}\frac{\Delta\rho}{\rho^{(1)}}\frac{1}{2}\frac{1}{U^{(1)}-U^{(2)}} \tag{3-62}$$

From the expression of D in (3-47),

$$\Delta D=\frac{k^2A_1^2}{4}\{\Delta\rho((U^{(2)}-c_{r0})^2-c_{i0}^2)-2\rho^{(1)}(U^{(2)}-c_{r0})\Delta c_r-2\rho^{(1)}c_{i0}\Delta c_i$$
$$+2\rho^{(1)}(U^{(1)}-c_{r0})\Delta c_r+2\rho^{(1)}c_{i0}\Delta c_i\}$$

By making use of (3-61) and (3-62) into the above relation,

$$\Delta D=-\frac{k^2A_1^2}{4}\Delta\rho\frac{(U^{(1)}-U^{(2)})^2}{2} \tag{3-63}$$

In the same way from the expression of E in (3-47),

$$\Delta E=g\Delta\rho-2k\{\Delta\rho(U^{(2)}-c_{r0})^2-2\rho^{(1)}\Delta c_r(U^{(2)}+U^{(1)}-2c_{r0})\}$$
$$+2kc_{i0}^2\Delta\rho+8kc_{i0}\rho^{(1)}\Delta c_i$$

Using (3-61) and (3-62),

$$\Delta E=-g\Delta\rho \tag{3-64}$$

and so at the limit $\Delta\rho\to 0$,

$$A_{21}=-\frac{D}{E}=-\frac{\Delta D}{\Delta E}=-\frac{k^2A_1^2}{2g}\frac{(U^{(1)}-U^{(2)})^2}{4}=-\frac{k^2A_1^2}{2g}c_{i0}^2 \quad (<0) \tag{3-65}$$

$$A_{22}=c_i\frac{B}{E}=c_i\frac{B}{\Delta E}=\frac{k^2A_1^2\rho^{(1)}}{g\Delta\rho}\frac{(U^{(1)}-U^{(2)})^2}{4}=\frac{k^2A_1^2\rho^{(1)}}{g\Delta\rho}c_{i0}^2 \tag{3-66}$$

By this way A_{21} has a finite negative value in proportion with c_{i0}^2, and A_{22} becomes infinitely large in inverse proportion with $\Delta\rho$. As $\Delta\rho$ is positive in the present treatment, A_{22} becomes positive infinity, and, if $\Delta\rho$ is negative, it tends to negative infinity.

When A_{22} is positive, the upper part of wave profiles deforms to windward in inverse to the direction of wave progress. When A_{22} is negative, the direction of deformation is leeward. The expression of (3-66) suggests that this deformation of wave profile may be treated as an initial value problem at the limit $\Delta\rho\to 0$, and so the gradual and explicit deformation of interfacial boundary, where the velocity of both fluids is discontinuous, may be found in this case. The direction of deformation of the interfacial boundary is controlled by the sign of $\Delta\rho$ in the present treatment of the second order approximation, and this may be compared with the analytical result of L. Rosenhead (1931), who used the condition of $\rho^{(1)}=\rho^{(2)}$.

Note : From the analysis of section 3.2, we can consider the Kelvin-Helmholtz instability as an extreme case of the amplification mechanism of interfacial waves by shear flows. The properties of the second order approximations of the Kelvin-Helmholtz instability cannot be concluded to be approved also in the general treatment of the amplifications of gravity waves by shear flows. But it seems a matter of care that the existence of A_{22} in section 3.3 at the case when $\rho^{(1)}$ is comparable to $\rho^{(2)}$ (apart from the problem of the sign of A_{22}) may become an important character for the development of interfacial waves under the shear flow.

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