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Evaluation of Resisting Moment Against Sliding as Rotating Mass

Two Dimensional Influence Charts of Stresses in a Semi-Infinite Elastic Body

by

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Contentr

Two Dimensional Influence Charts of Stresses in a Semi-Infinite Elastic Body

CONTENTS

半無限弾性体内の応力を求める二次元的影響図表

沢　口　正　俊

概　　要

地盤上に任意の断面を持つ堤防によって生ずる垂直応力を，地盤を半無限弾性体と仮定した場合に，二次元的な影響図表を使って近似的ではあるが，非常に迅速に求めた。又同じような水平応力およびセン断応力を求める影響図表を使えば，スケンプトンの式を使って地盤中の初期間隙水圧が求まる。この間隙水圧が二次元的に散逸することを二次圧密理論の近似解法によって説明した。最後に地盤中の二つの応力が重ね合った時の応力状態を幾何学的に調べた。

Two Dimensional Influence Chart of Stresses in a Semi-Infinite Elastic Body

MASATOSHI SAWAGUCHI*

SYNOPSIS

This report states that the vertical stress due to an embarkment of any shape of cross-section on the ground assumed to be elastic could be obtained approximately but very easily by using a two dimensional influence chart. And, in appendix, it was described that, with influence charts with respect to horizontal and shearing stresses, this could be applied to obtain the initial pore-water pressure related to the consolidation. This possibility was developed by relaxation method of two dimensional consolidation and, as one example, was illustrated by the case of a line load application, and, furthermore, the possibility of superposing the stresses due to such many line loads was also investigated geometrically.

§ 1. Introduction

In such construction works as roads, rivers or harbors, there are many cases that what magnitude of stress will be occured in the foundation under such structures as have the same cross-section along almost infinite length must be investigated. But, as it is rather difficult to obtain the solution based upon the elastic theory, an approximate answer is sometimes urgent.

Up to this time, in order to obtain the stress occured in the ground under such a structure of finite length, influence charts have been used widely. However, in case of an infinite strip load of which the cross-section is not uniform, for example, an embankment, it seems easier and more accurate to obtain the stress by using a two dimensional influence chart which will be stated in this report because stress condition in such a case is plain strain. Therefore, the two dimensional influence chart is recommended to be applied to any shape of load section to obtain a relatively accurate answer as regards plain strain problem.

Then, in order to check the accuracy of this influence chart, this chart was applied to an example which was given by Osterberg in his report. And, furthermore, how much ratio of length to width of a load area with uniform weight can be considered as two dimensional problem within a range of negligible error, even if it has

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a finite length of site, was investigated by comparing the value obtained by this influence chart with that one by analytical procedure.

§ 2. Survey of the literature

To obtain the stress by elastic theory in foundation under a structure of a complicated shape of site, many works have been pursued by using influence charts so far. In 1938, Burmister devised a chart for stress by using a concept of influence line and graphycal calculation method, and he stated how to obtain easily the stress at a point in foundation under a structure by counting the number of blocks in his chart which are occupied by the trace of the site. But this procedure needs a separate chart for a given depth, and so, it was not very convenient to obtain the stress at any depth.

In the discussion of the same report, N. M. Newmark suggested his new chart which gives a value of dimensionless coefficient as regards the stress based upon Boussinesq's elastic equation. As this coefficient includes only dimensionless term, that is, the ratio of horizontal distance between a point in question and a point where a load is applied, to the depth, the stress at any depth in foundation can be obtained by only one sheet of chart.

In his report published in 1940, he discussed again about his chart and presented a more improved one, where he first named the chart for an influence chart and the value of stress which one block indicates, for influence value which is constant in his chart.

In 1942, Newmark published further the latest influence charts for horizontal stress, shearing stress and the sum of principal stresses in elastic mass.

Recently, Fenske made out the same kind of a chart based upon Westergaard's equation as that one by Newmark.

§ 3. Two dimensional influence chart

When a structure is constructed on the ground, the stress occurs in the foundation which causes settlement or shearing failure of the soil. Therefore, the designers are demanded to know what magnitude of stress will be occured in the ground under a structure in advance. In this case, it is most easy and practicable to estimate the stress by applying elastic theory to the stress in the ground which is supposed to be semi-infinite mass.

A load put on the ground gives rise to stress to any direction in the ground. Especially, the stress in vertical direction is the more interesting because it is applied to the problem of settlement. The equation for vertical stress in semi-infinite elastic mass under a point load vertical to the boundary are given by both Boussinesq and Westergaard respectively in different expression as follows;

Boussinesq
$$\sigma_z = \frac{3Q}{2\pi} \frac{z^3}{(r^2+z^2)^{5/2}} \tag{1}$$

Westergaard
$$\sigma_z = \frac{Qz}{2\pi} \frac{\eta}{(\eta^2 z^2 + r^2)^{3/2}} \tag{2}$$

where σ_z is vertical stress, Q a point load, z the depth, r the horizontal distanse and

$$\eta = \sqrt{\frac{1-2\mu}{2-2\mu}}$$

In Boussinesq's equation, it is assumed that the mass is isotropic and homogeneous, and in Westergaard's equation, it is assumed that the strain parallel to the boundary surface is zero.

When such a complicated distributed load as an actural structure is constructed on the ground, these equations can be superposed by integrating the stress under a point load, because the principle of superposition is valid in elastic theory.

In the case that a structure has a site of almost infinite length in comparision with its width, the vertical stress can be calculated by Boussinesq's equation as follows; the vertical stress under a uniform line load of unit length of weight, p, can be given by integrating eq. (1) as

$$\sigma_z = \frac{2p}{\pi z} \frac{1}{\left[1+\left(\frac{r}{z}\right)^2\right]^2} \tag{3}$$

Supposing that this line load is distributed between x_1 and x_2, which are the distance along axis in the direction perpendicular to the line load, eq. (3) becomes

$$\sigma_z = \frac{C}{2\pi} \tag{4}$$

where $C = (2\theta_2 + \sin 2\theta_2) - (2\theta_1 + \sin 2\theta_1)$

and $\frac{x_1}{z} = \tan\theta_1 \qquad \frac{x_2}{z} = \tan\theta_2$

From the equations above we can obtain vertical stress at any depth under an infinite strip load of uniform weight, p, which is distributed between x_1 and x_2. If scale is established along x axis so that σ_z may have a constant value — we can name it for influence value like Newmark's — that is, C has a constant value, the vertical stress at any depth under the origin of the scale due to an infinite strip load of uniform weight distributed on every each scale becomes constant.

Accordingly, the stress due to any shape of load section can be estimated by summing up the number of the areas occupied on that scale by the trace of load section. In that case, we must give such a reduction to the dimension of the load section in order to make the trace as the distance to the depth at question corresponds to the scale distance, and then we put the reduced trace in dimension on the scale so that the point in question may be located on the ordinate. Thus the number of the scale which is occupied by the traced area of the load section gives the stress in question with the reduction ratio multiplied by which the quantity of load is reduced so that some magnitude, for example, the peak of load coincides with unit on the ordinate.

Now, let us make out an influence chart with some value and obtain the stress under an embankment as an example by using this chart. For example, supposing the influence value to be 0.025, from eq. (4)

$$0.025=\frac{C}{2\pi}$$

$$C=0.157$$

To determine the scale along x axis beginning at the origin, the value x_2/z is first determined by substitution of $x_1=0$ into eq. (5). Again substituting the value x_2/z for the value x_1/z into the same equation, the value x_2/z can be determined. Thus, by repeating this procedure, a series of the values x/z can be determined. Here, if we take a constant as the scale distance, for example, 10cm in this case, the series of x values can be plotted along x axis as a scale. As a result, a uniform strip load which is distributed in each scale induces a same value of vertical stress at given depth in the ground. The two dimensional influence chart is given in Fig. 1, where, for $x/z>1.827$, the influence value is reduced to 0.0025 so that the effect of load at a great distance on the vertical stress may be accurately estimated. This scale lines are written by the dotted lines in the chart.

Now, let's obtain the vertical stress in the ground under one of the typical embankments which Osterberg gave in his report, by using this influence chart. The cross-section of this embankment is a trapezoid as shown in Fig. 2. After reducing the dimensions of the embankment so that the distance down to the depth, 20′, in question may corresponds to the scale distance in the chart, that is, 10cm, the cross-section of the embankment is traced on the chart. As the scale along x axis is symmetric with respect to the origin, only one side of the scale is shown in the figure. "Left Slope" in the chart means the left side of slopes of the embankment and "Right Slope" means the opposite. In this example,

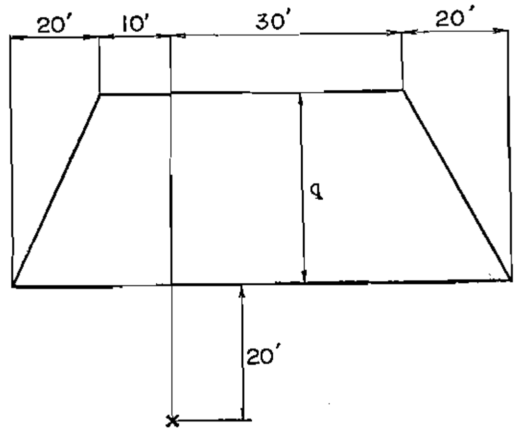

Fig. 2.

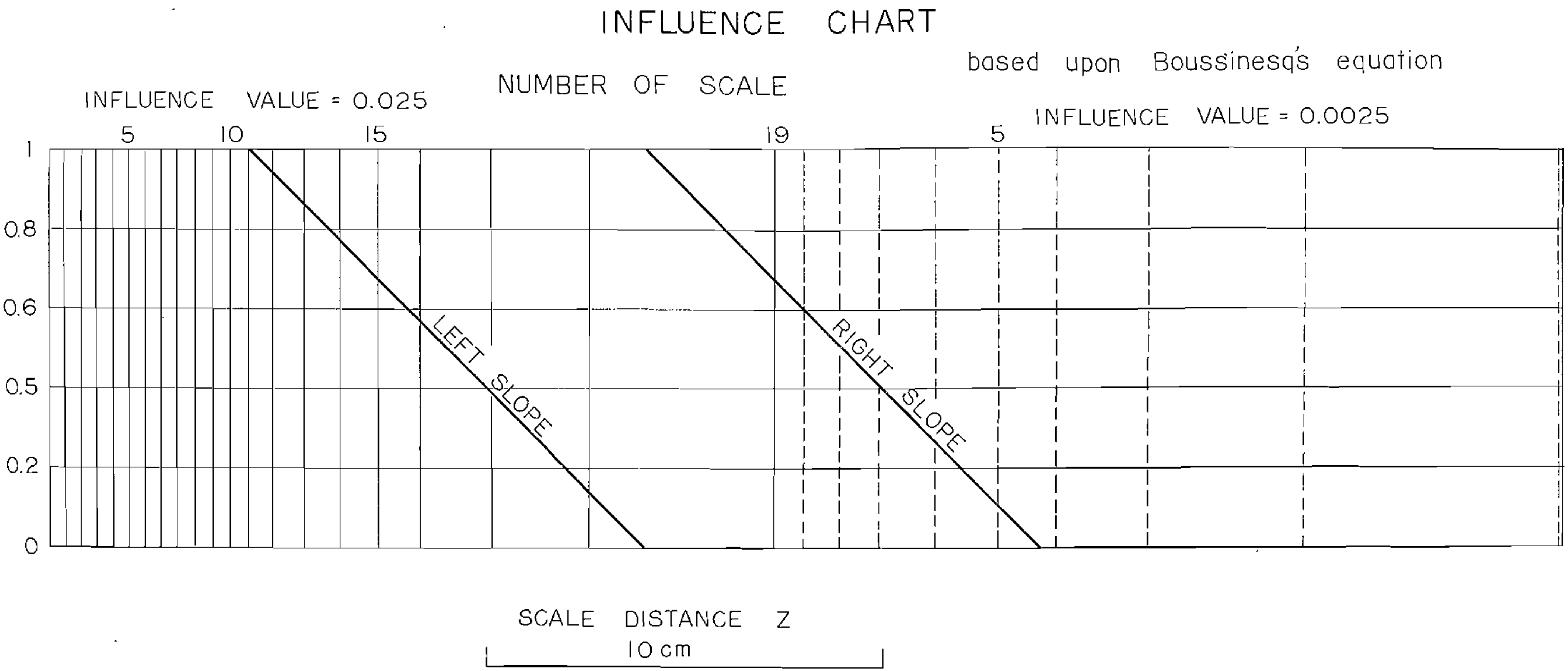
INFLUENCE CHART
based upon Boussinesq's equation
NUMBER OF SCALE
INFLUENCE VALUE = 0.025
INFLUENCE VALUE = 0.0025
5
10
15
19
5
1
0.8
0.6
0.5
0.2
0
LEFT SLOPE
RIGHT SLOPE
SCALE DISTANCE Z
10 cm

Fig. 1

as the maximum load is q, the cross-section is traced so that this maximum corresponds to the unit on the ordinate. Thus, the number of scales including the fractions of scale which are occupied by the traced area becomes 34.55 for influence value, 0.025, and 2.15 for 0.0025.

Accordingly, the vertical stress at the point in question, σ_z, is

$$(0.025\times34.55+0.0025\times2.15)=0.869$$

If it is demanded to take more accurately the fractions in the part of scales which are occupied by the straight slope line, what follows is recommended. Supposing the distance of the scale lines which are cut by the slope to be l_1, l_2, l_3 and l_4 which correspond to the scale on the ordinate as shown in Fig. 3, the total fraction in that part is obtained as follows;

$$\frac{1}{2}(l_1+2l_2+2l_3+l_4) \qquad (6)$$

The triangular in the most remote part of scale cannot be estimated without by inspection. Comparing this result with the value given by Osterberg, there is only about 1 % of error in this value as shown in Tab. 1, but its accuracy seems to be good enough for practical purpose.

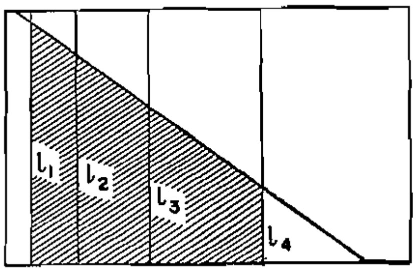

Fig 3.

Next, let's consider the influence chart based upon Westergaard's equation. This case also begins with the integration to a line load just like Boussinesq's case. As Westergaard's equation is given in eq. (2), the vertical stress under a line load is obtained

Tab. 1

			Westergaard		
	Osterberg	Boussinesq	$\mu=0$	$\mu=0.25$	$\mu=0.33$
σ_z	0.875 q	0.869 q	0.695 q	0 730 q	0.774 q

by integrating this equation with respect to x within infinite range, that is,

$$\sigma_z=\frac{\eta p}{\pi z}\frac{1}{\eta^2+\left(\frac{z}{x}\right)^2} \qquad (7)$$

Furthermore, supposing a uniform strip load distributes between x_1 and x_2, the vertical stress at a given point is given by integrating this equation with respect to x as follows,

$$\sigma_z=\frac{p}{\pi}\left(\tan^{-1}\frac{x_2}{\eta z}-\tan^{-1}\frac{x_1}{\eta z}\right) \quad (8)$$

Likewise as in case of Boussinesq's equation, a two dimensional influence chart can be made out from this equation. In this case also, the influence value can be provided by fixing a constant value as the vertical stress in the ground under a uniform strip load which is distributed in each scale along x axis. The completed chart is shown in Fig. 4. Here again, we can obtain the vertical stress in the same example as in case of Boussinesq's equation by using this chart, where $\mu=0$, 0.25 and 0.33. This result also is added in Tab. 1. As shown in the table, the value for $\mu=0.33$ is the most close to the value obtained by the influence chart based upon Boussinesq's equation. And also, as Taylor stated in his book, the value for $\mu=0$ is about two thirds of that by Boussinesq's equation.

As it seems reasonable that the vertical stress in the ground near by the center of structure of which the ratio of length to its width is almost infinite can be a two dimensional problem without practical error, so, in order to show how much ratio of length to its width can be considered to be a two dimensional problem in such a case, the stress occured under the structure as shown in Fig. 5 was compared in case of $\mu=0$ by using both the two dimensional influence chart and the analytical method presented in Taylor's book.

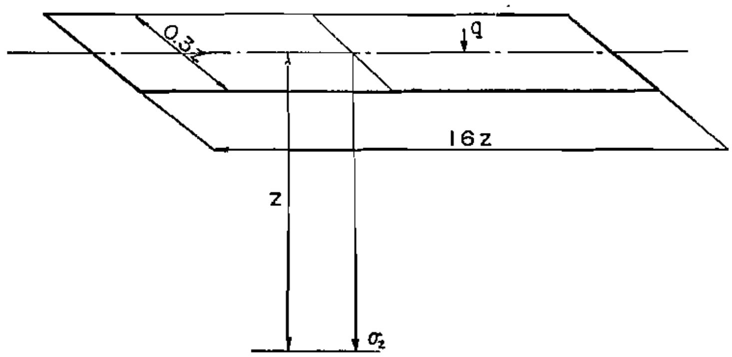

Fig. 5.

This was done for the case of a uniform load and the vertical stress at some depth right under the center of the structure was obtained. This comparison is shown in Table 2. As a result, it is clear that the vertical stress obtained from the influence chart based upon Boussinesq's equation is almost the same as the value by the analytical method; that is, the value by that based upon Westergaard's equation differs the analytical value by only 3 %.

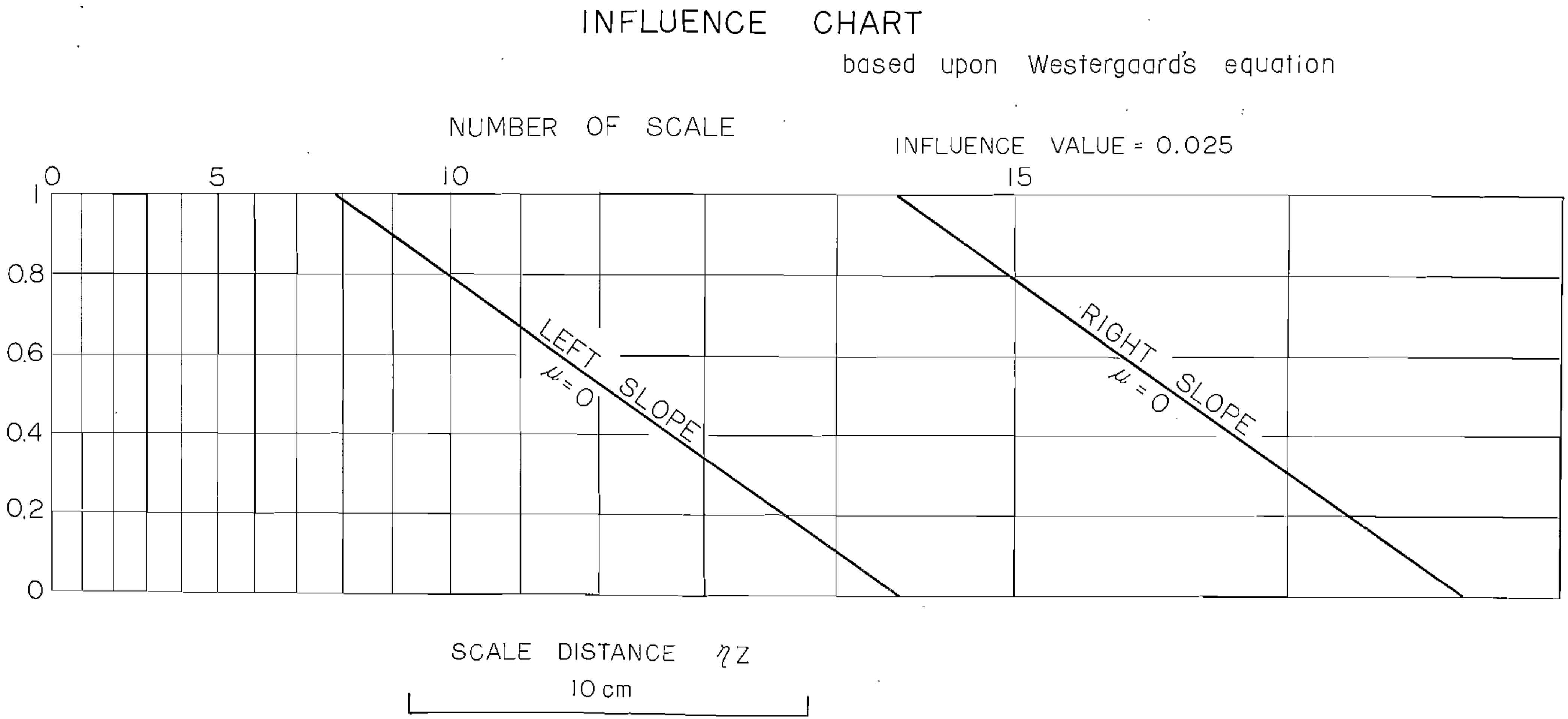
INFLUENCE CHART
based upon Westergaard's equation
NUMBER OF SCALE
INFLUENCE VALUE = 0.025
0
5
10
15
1
0.8
0.6
0.4
0.2
0
LEFT SLOPE
μ = 0
RIGHT SLOPE
μ = 0
SCALE DISTANCE ηz
10 cm

Fig. 4

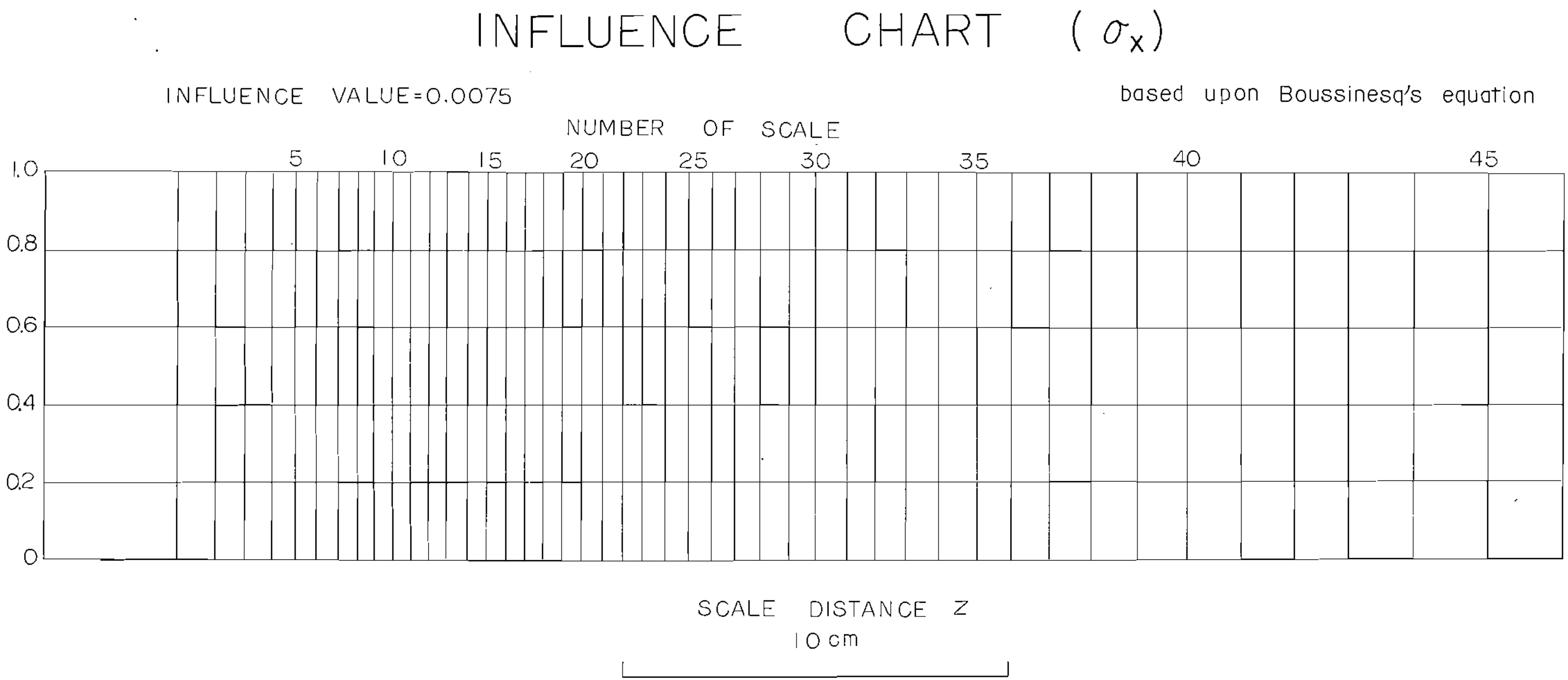
INFLUENCE CHART (σ_x)
INFLUENCE VALUE=0.0075
based upon Boussinesq's equation
NUMBER OF SCALE
5
10
15
20
25
30
35
40
45
1.0
0.8
0.6
0.4
0.2
0
SCALE DISTANCE Z
10 cm

Fig. 6

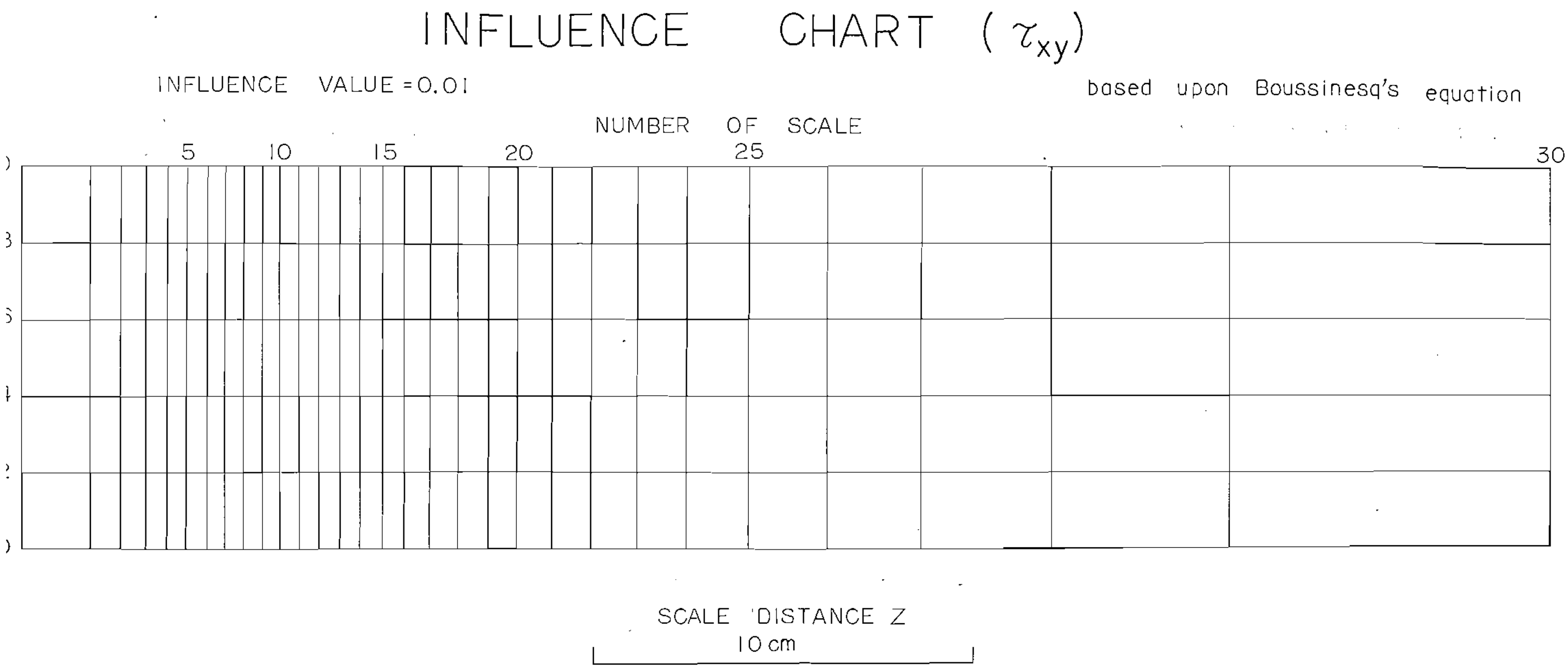
INFLUENCE CHART (τ_{xy})
INFLUENCE VALUE = 0.01
based upon Boussinesq's equation
NUMBER OF SCALE
5
10
15
20
25
30
SCALE DISTANCE Z
10 cm

Fig. 7

Tab. 2

	Boussinesq	Westergaard $\mu = 0$
Taylor	0.134 q	0.180 q
auther	0.134 q	0.185 q

Therefore, this influence chart is valid enough for practical purposes even in case of a structure whose ratio of the length to its width is considerably large.

§ 4. Conclusion

While the influence chart which was made out by Newmark is valid to obtain the vertical stress in the ground under a structure of a limited site, the two dimensional influence chart which was made out by the auther is valid in case of an infinite or almost infinite strip load, whatever the cross-section of the load may be. For example, the validity of this chart will be demonstrated in a problem such as in settlement of an embankment including the weight due to other structures, for example, a breakwater or pavement.

In many case of two dimensional stress problem where more accuracy is demanded, the analytical procedure should be used, but this influence chart, just likewise as other influence charts, is useful to obtain an approximate value very quickly.

Reference

1) N. M. Newmark (1942): Influence Charts for Computation of stresses in Elastic Foundation. Bulletin No. 338, Engr. Exp. Station, University of Illinois.
2) J. O. Osterberg: Influence Values for Vertical Stresses in a Semi-Infinite Mass due to an Embankment Loading. Proc. of Int. of Soil Mech. and Found. Engr. 4—1
3) D. W. Taylor: Fundamentals of Soil Mechanics. John Wiley & Sons.
4) D. M. Burmister (1938) : Graphical Distribution of Vertical Pressure Beneath Foundations. Trans. of ASCE Vol. 103
5) C. W. Fenske : Circular No. 21 The Univ. of Texas, Bureau of Engineering Research.
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APPENDIX

(A) Approximate calculation of the two dimensional consolidation by graphical procedure

Just likewise as in the case of σ_z, the influence charts with respect to σ_x and τ_{xz} based upon Boussinesq's equations can be also made up. These charts are shown in Fig. 6 and Fig. 7.

With all these equations, we can find the principal stresses occured at a given point in the ground due to a load of which the form of the cross-section is arbitrary. Therefore, if the relation between the principal stresses and pore-water pressure is defined, for example, as

$$\Delta u = B[\Delta\sigma_3 + A(\Delta\sigma_1 - \Delta\sigma_3)] \qquad (9)$$

which is called for Skempton's formula, we can estimate the pore-water pressure at the point in the ground due to the load. As time goes on, the pore-water pressure will dissipate. The degree of dissipation of the pore-water pressure can be estimated by the two dimensional consolidation theory. This estimation also can be done with the approximation by the graphical solution; that is, in imitation of the approximation by the graphical solution in one dimensional consolidation theory by G. de Josselin de Jong, the author applied the same method to the two dimensional case. As the equation of two dimensional consolidation in the case of homogeneous soil mass is expressed as

$$\frac{\partial u}{\partial t} = C_v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (10)$$

so, if squares mesh of small size, $\Delta x = \Delta y = \Delta l$, is drawn in the ground parallel to the direction of x axis and y axis, and the pore-water pressure at some node, 0, at any time is supposed to be u_0,

$$\frac{u_0' - u_0}{\Delta t} = \frac{C_v}{(\Delta l)^2}(u_1 + u_2 + u_3 + u_4 - 4u_0)$$

where u_0' is u_0 after the lapse Δt, and u_1, u_2, u_3 and u_4 are the pore-pressure at the neighbor gride as shown in Fig. 8. C_v is a coefficient of consolidation.

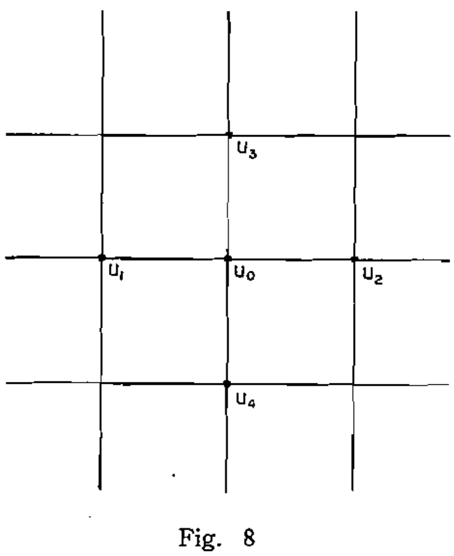

Fig. 8

As Δt and Δl can be arbitrarily given independently, if Δt is defined so that $\Delta t = \frac{(\Delta l)^2}{4C_v}$, u_0' is expressed as follows;

$$u_0'=\frac{1}{4}(u_1+u_2+u_3+u_4) \qquad (11)$$

This simple calculation may be begun at an arbitrary grid. After one cycle of calculation ended, these figures show the pore-water pressure after the lapse, Δt. Repeating this procedure, the pore-water pressure in the ground after the lapse of any time can be obtained. Fig. 9 shows one example of a line load application on the ground. In this example, the principle stresses can be expressed as

$$\Delta\sigma_1=\frac{2P}{\pi r}\cos\theta \qquad \Delta\sigma_3=0$$

If the magnitude of a line load is $\pi/2A$,

$$\Delta\sigma_1=\frac{1}{Ar}\cos\theta$$

and then

$$\Delta u^*=\frac{1}{r}\cos\theta \qquad (12)$$

The computation of the approximation was repeated eight times, the mesh size being unity, and the values of the pore-water pressure were written down in each square. We will call temporarily these values "pore-water pressure value".

Now, let us apply this pore-water pressure value to the practical case. If the magnitude of an applied load and the mesh size in-situ are supposed to be P and H, the pore-water pressure in-situ, Δu, is written by

$$\Delta u=\left(\frac{2A}{\pi}\right)\frac{P}{H}\times\Delta u^* \qquad (13)$$

where Δu^* is the pore-water pressure value at the grid corresponding to the position required for pore-water pressure in-situ. The pore-water pressure there after the lapse, T, can be obtained from the pore-water pressure values at N cycle of computation. N is determined from

$$N=\frac{T}{\Delta t} \qquad (14)$$

$$\Delta t=\frac{H^2}{4C_v} \qquad (15)$$

where the dimensions are as follows;

$P=\pi/2\ A$

0	0	0	0	0	0	0	0	0
0.514 0.400 0.247 0.222 0.163 0.153 0.122 0.116	1.000 0.382 0.333 0.217 0.199 0.149 0.141 0.114	0.514 0.400 0.247 0.222 0.183 0.153 0.122 0.116	0.199 0.216 0.186 0.156 0.138 0.114 0.109 0.092	0.100 0.102 0.107 0.099 0.091 0.086 0.077 0.074	0.059 0.059 0.060 0.061 0.059 0.057 0.055 0.052	0.039 0.039 0.038 0.038 0.039 0.038 0.038 0.037	0.026 0.027 0.027 0.027 0.026	0.020 0.019
0.399 0.391 0.369 0.285 0.275 0.225 0.216 0.183	0.500 0.532 0.375 0.354 0.273 0.261 0.214 0.207	0.399 0.391 0.369 0.285 0.275 0.225 0.216 0.183	0.249 0.245 0.241 0.233 0.203 0.186 0.171 0.165	0.153 0.154 0.153 0.152 0.149 0.139 0.135 0.125	0.100 0.100 0.100 0.100 0.100 0.099 0.095 0.094	0.066 0.068 0.068 0.068 0.067	0.050 0.050 0.050	
0.300 0.299 0.298 0.294 0.262 0.257 0.228 0.223	0.333 0.337 0.345 0.305 0.298 0.280 0.254 0.224	0.300 0.299 0.298 0.294 0.262 0.257 0.229 0.223	0.230 0.229 0.227 0.226 0.222 0.206 0.202 0.185	0.167 0.166 0.164 0.164 0.163 0.162 0.155 0.153	0.120 0.120 0.119 0.119 0.116 0.118 0.115	0.088 0.088 0.088 0.087	0.067 0.066	
0.235 0.235 0.235 0.234 0.234 0.223 0.221 0.208	0.250 0.250 0.251 0.253 0.243 0.241 0.225 0.223	0.235 0.235 0.235 0.234 0.234 0.223 0.221 0.208	0.200 0.109 0.198 0.198 0.196 0.169 0.186	0.160 0.159 0.159 0.158 0.158 0.157 0.153	0.125 0.125 0.124 0.124 0.124 0.123	0.098 0.097	0.077 0.077	
0.192 0.192 0.192 0.192	0.200 0.200 0.200 0.198 0.198 0.194	0.192 0.192 0.192 0.192 0.188	0.172 0.172 0.169	0.147 0.147 0.146	0.122 0.122	0.100	0.082	
0.162	0.161	0.162	0.150	0.133	0.115	0.098		
0.140	0.143	0.140	0.132	0.121				

Fig. 9

P (t/m) H (m) C_v (m/day)

Let us obtain the pore-water pressure at the position, F, in Fig. 10.

Here $X=10$ m $Y=20$ m $P=10$ t/m $T=100$ days

$C_v=1.0\times10^{-2}$m²/day

Supposing $A=1/3$, from eq. (14) and eq. (15),

$$\Delta t=\frac{2^2}{4\times10^{-2}}=100 \text{ days}$$

$$N=\frac{100}{100}=1$$

Therefore, we can use the pore-water pressure value at one cycle of computation. Taking, for example, $H=10$ m, from the value in the figure, $\Delta u^*=0.391$

In a result, from eq. (13),

$$\Delta u=\frac{2\times0.33\times10}{3.14\times10}\times0.391=0.082 \text{ t/m}^2$$

The case of an arbitrary cross-section of an embankment load will be demonstrated later.

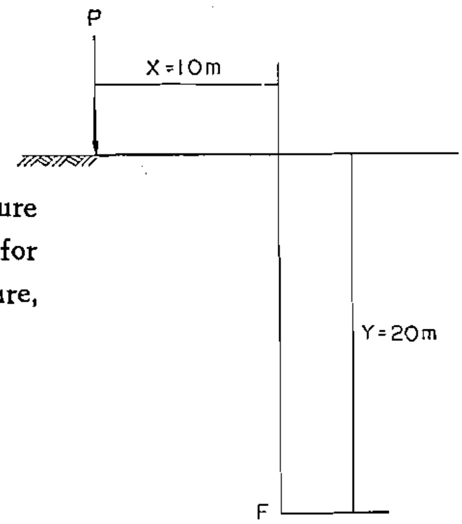

Fig. 10

(B) Superposition of stresses

If some stress state is combined with another stress state, the magnitude and the direction of the conbined principal stresses will differ from the previous state. Supposing that the pore-water pressure can be determined by the magnitude of the principal stresses, we have first to know the magnitude of principal stresses in order to obtain the pore-water pressure in the ground. For this purpose, the auther has made out the method of combining more than two stress states. Now, let us consider the most simple case of combining two stress states. Supposing that the former stress state is signed as (A) and the superposed stress state is (B), and that the directions of the both principal stresses intersect each other by the angle, α, and supposing that the direction of the combined principal stress rotates from the direction of the principal stress in the stress state (A) by the angle, θ, the stresses of the states (A) and (B) to that direction become respectively,

$$
\left.\begin{aligned}
&\left.\begin{aligned}
\alpha &= \frac{\sigma_1+\sigma_2}{2}+\frac{\sigma_1-\sigma_2}{2}\cos 2\theta \\
\tau &= \frac{\sigma_1-\sigma_2}{2}\sin 2\theta
\end{aligned}\right\} \quad (\mathrm{A}) \\
&\left.\begin{aligned}
\sigma' &= \frac{\sigma_1'+\sigma_2'}{2}+\frac{\sigma_1'-\sigma_2'}{2}\cos(2\theta-2\alpha) \\
\tau' &= \frac{\sigma_1'-\sigma_2'}{2}\sin(2\theta-2\alpha)
\end{aligned}\right\} \quad (\mathrm{B})
\end{aligned}\right. \qquad (16)
$$

where the apostrophe means the stresses in the state (B). Other principal stresses intersecting with the above principal stresses by a right angle can be likewise treated by substituting $2\theta+\pi$ for 2θ. Expressing the combined stresses for σ'' and τ'', these are;

$$
\begin{aligned}
\sigma'' &= \frac{\sigma_1+\sigma_2}{2}+\frac{\sigma_1'+\sigma_2'}{2}+\frac{\sigma_1-\sigma_2}{2}\cos 2\theta+\frac{\sigma_1'-\sigma_2'}{2}\cos(2\theta-2\alpha) \\
\tau'' &= \frac{\sigma_1-\sigma_2}{2}\sin 2\theta+\frac{\sigma_1'-\sigma_2'}{2}\sin(2\theta-2\alpha)
\end{aligned} \qquad (17)
$$

However, this direction becoming that of the combined principal stress requires the condition that τ'' must be zero.

$$\frac{\sigma_1-\sigma_2}{2}\sin 2\theta_0+\frac{\sigma_1'-\sigma_2'}{2}\sin(2\theta_0-2\alpha)=0$$

Rewriting this equation,

$$\tan 2\theta_0=\frac{\dfrac{\sigma_1'-\sigma_2'}{2}\sin 2\alpha}{\dfrac{\sigma_1-\sigma_2}{2}+\dfrac{\sigma_1'-\sigma_2'}{2}\cos 2\alpha} \qquad (18)$$

This angle, $2\theta_0$, can be obtained easily by graphical procedure as follows. In Fig. 11, taking $\frac{\sigma_1-\sigma_2}{2}$ as the distance, *AM*, and $\frac{\sigma_1'-\sigma_2'}{2}$ as the distance, *BM*, so that the angle between *AM* and *BM* becomes 2α, the angle, $2\theta_0$, is obtained as the angle, *MAB*. This rule can be verified as follows. Drawing a perpendicular passing through the point, *B*, down to the line, *AM*, and denominating the intersection as *B'*,

$$\tan\ (\text{the angle } BAM)=\frac{BB'}{AB'}=\frac{BB'}{AM+MB'}$$

$$=\frac{\dfrac{\sigma_1'-\sigma_2'}{2}\sin 2\alpha}{\dfrac{\sigma_1-\sigma_2}{2}+\dfrac{\sigma_1'-\sigma_2'}{2}\cos 2\alpha}$$

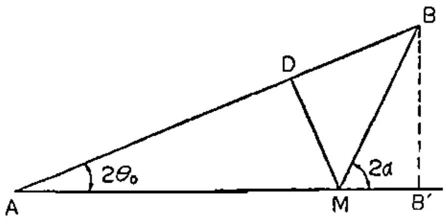

Fig. 11

The magnitude of the combined principal stress is obtained by adding the length of *AB* to $\frac{\sigma_1+\sigma_2}{2}+\frac{\sigma_1'+\sigma_2'}{2}$.

This is resulted from the fact that, writing a perpendicular, *MD*, to the line, *AB*, passing through the point, M,

$$AB=AD+DB=\frac{\sigma_1-\sigma_2}{2}\cos 2\theta+\frac{\sigma_1'-\sigma_2'}{2}\cos(2\theta-2\alpha)$$

This length is equal to the difference between σ'' and $\frac{\sigma_1+\sigma_2}{2}+\frac{\sigma_1'+\sigma_2'}{2}$. Now, let us consider the stresses to a given direction. Taking 2θ as the angle between the direction required and the direction of the principal stress state (A), the stresses to the direction required can be obtained as follows; that is, drawing a line intersecting with the line, AM, by the angle, 2θ, passing through the point, A, and drawing a perpendicular, BE, to that line passing through the point, B, as shown in Fig. 12,

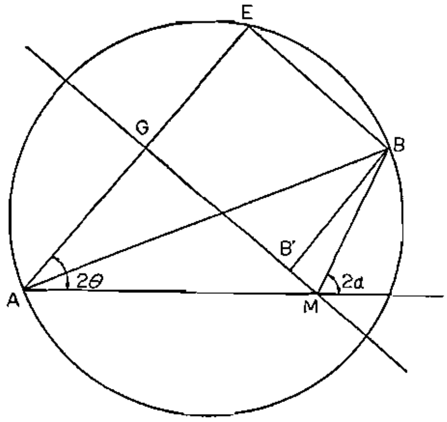

Fig. 12

$$\sigma''=AE+\frac{\sigma_1+\sigma_2}{2}+\frac{\sigma_1'+\sigma_2'}{2}$$

$$\tau''=BE$$

This can be verified as follows. Writing a perpendicular, MG, to the line, AE, passing through the point, M,

$$AG=\frac{\sigma_1-\sigma_2}{2}\cos 2\theta \qquad EG=\frac{\sigma_1'-\sigma_2'}{2}\cos(2\theta-2\alpha)$$

and

$$MG=\frac{\sigma_1-\sigma_2}{2}\sin 2\theta \qquad B'M=\frac{\sigma_1'-\sigma_2'}{2}\sin(2\theta-2\alpha)$$

Therefore,

$$\sigma''=\frac{\sigma_1+\sigma_2}{2}+\frac{\sigma_1'+\sigma_2'}{2}+\frac{\sigma_1-\sigma_2}{2}\cos 2\theta+\frac{\sigma_1'-\sigma_2'}{2}\cos(2\theta-2\alpha)$$

$$\tau''=\frac{\sigma_1-\sigma_2}{2}\sin 2\theta+\frac{\sigma_1'-\sigma_2'}{2}\sin(2\theta-2\alpha)$$

This means that any point on the circle whose diameter is the length, AB, is to express the combined stress state, and the magnitude of the stresses can be shown by the distance between the point, M, and the point, A or B, and further, that the combined stress state can be expressed by a Mohr's circle.

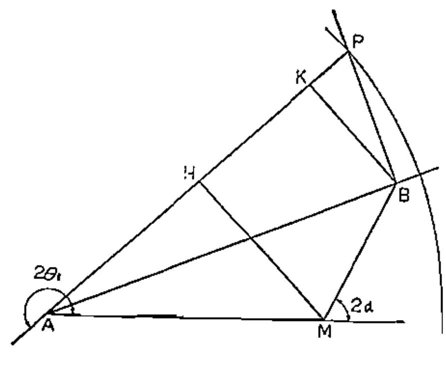

Fig. 13

Next, let us consider what is the magnitude of the principal stresses, σ'' and τ'', and which is the direction of the principal stress at failure condition. These values are obtained by graphical procedure as follows ; in Fig. 13, a perpendicular, BP, to the line, AB, passing through the point, B, is drawn and a circle of which the diameter is $\frac{\sigma_1+\sigma_2}{2}+\frac{\sigma_1'+\sigma_2'}{2}$ is drawn around the center, A. The circle intersects with the line, PB, at the point, P. Then, after drawing a line passing through the points, A, and P, the angle between the direction of the principal stress in the state (A) and the direction of the failure plane in the combined stress state can be obtained by the complementary angle against the angle, MAP. This will be verified in the following. When the combined stress state is at failure, τ''/σ'' should have the maximum value; that is, the equation

$$\frac{\tau''}{\sigma''}=\frac{\frac{\sigma_1-\sigma_2}{2}\sin 2\theta+\frac{\sigma_1'-\sigma_2'}{2}\sin(2\theta-2\alpha)}{\frac{\sigma_1+\sigma_2}{2}+\frac{\sigma_1'+\sigma_2'}{2}+\frac{\sigma_1-\sigma_2}{2}\cos 2\theta+\frac{\sigma_1'-\sigma_2'}{2}\cos(2\theta-2\alpha)}$$

has to become at the maximum. This calculation can be carried out by differentiating this equation by the angle, θ, and making the derivative zero. The result is

$$\left(\frac{\sigma_1+\sigma_2}{2}+\frac{\sigma_1'+\sigma_2'}{2}\right)\left(\frac{\sigma_1-\sigma_2}{2}\cos 2\theta_f+\frac{\sigma_1'-\sigma_2'}{2}\cos(2\theta_f-2\alpha)\right)$$
$$+\left(\frac{\sigma_1-\sigma_2}{2}\right)^2+\left(\frac{\sigma_1'-\sigma_2'}{2}\right)^2+2\left(\frac{\sigma_1-\sigma_2}{2}\right)\left(\frac{\sigma_1'-\sigma_2'}{2}\right)\cos 2\alpha=0 \qquad (19)$$

The angle, θ_f, can be obtained by the graphical procedure as follows. If perpendiculars, MH, and BK, to the line, AP, passing through the points, M, and B, respectively are drawn, and these lines intersect at the point, H, and K, respectively with the line, AP,

$$AB^2=AM^2+BM-2AM\ BM\cos(\pi-2\alpha)$$
$$=\left(\frac{\sigma_1-\sigma_2}{2}\right)^2+\left(\frac{\sigma_1'-\sigma_2'}{2}\right)^2+2\left(\frac{\sigma_1-\sigma_2}{2}\right)\left(\frac{\sigma_1'-\sigma_2'}{2}\right)\cos 2\alpha$$

whiles, $$AP=\frac{\sigma_1+\sigma_2}{2}+\frac{\sigma_1'+\sigma_2'}{2}$$

$$AK=AM\cos(2\theta_f-\pi)+BM\cos(2\alpha-2\theta_f+\pi)$$
$$=\frac{\sigma_1-\sigma_2}{2}\cos 2\theta_f+\frac{\sigma_1'-\sigma_2'}{2}\cos(2\theta_f-2\alpha)$$

and, as $AB^2=AP\cdot AK$, eq (19) is found to be satisfied. Therefore, the combined stresses at failure plane can be obtained by substituting the angle, θ_f, into eq. (17). When cohesion must be considered, it is enough to take $\frac{\sigma_1+\sigma_2}{2}+\frac{\sigma_1'+\sigma_2'}{2}+C\cot\phi$ as the length of the diameter, AP. The problem about combining more than three stress states can be solved by carring on the above-mentioned procedure one by one.

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