REPORT

OF

PORT AND HARBOUR TECHNICAL RESEARCH INSTITUTE

REPORT NO 1

Behavior of Tensile and Flexural Cracks in Reinforced Concrete Members

by
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February 1963

PORT AND HARBOUR TECHNICAL RESEARCH INSTITUTE
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162 Kawama Yokosuka-City Kanagawa-Pretecture, Japan

Errata of

Report of Port and Harbour Technical Research institute **REPORT NO 1**

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BEHAVIOR OE TENSILE AND FLEXURAL CRACKS IN REINFORCED CONCLETE MEMBERS

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Contents

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Introduction

The use of high-tensile-strength steel and deformed bars for reinforcing bars is an increasing tendency. The former increases the working stresses considerably, accordingly decreasing the required amount of reinforcement. The latter, generally, eliminates the need for end hooks, which results in an equivalent economical effect as the former. On the use of them, however, lacks of an exact understanding of the behavior of cracking may results in wider cracks or serious failure. Especially in harbour structures, cracks in reinforced concrete members should be carefully considered, no matter they may be tensile or flexural cracks. Unfortunately in this field, the standard procedures or specifications for evaluation or limitation of cracking in reinforced concrete members have not yet been established except in few countries. To expect the economical and perfect use of high-tensile-strength steel or deformed bars, it is considered of urgent need to establish them through extensive theoretical and experimental studies and comprehensive field investigations.

1. Purpose and Scope

The principal object of this study is to review the present theories and hypotheses about the cracking in the reinforced concrete and to discuss the effects of cracking on the stress distribution in reinforcing steel bars. Considering the situations, under which cracking takes place, cracks in reinforced concrete may be classified into four groups, i. e., cracks under pure tension, pure bending, pure shearing, and combined stress conditions of these former three. Probably one more group may be added, which is caused by some chemical reactions as alkali-aggregates reactions. The discussion in this study will be limited to the first two groups, because the third and fourth can be analyzed with the same analogy as in the former two with some interpretation and the last is not defined clearly and also includes too much complicated stress conditions to be analyzed theoretically.

In the most of prevailing theories on cracking, the following assumptions are made to simplify the problem:

(1) The stress in reinforcement does not exceed the proportional limit.

(2) All reinforcement bars in a given member are continuous through out their length and are of the same size and shape.

(3) The strain in the concrete are proportional to the stress.

(4) The effects of shearing deformation on the spacing and the width of cracks are neglesible in a case of pure tension.

(5) There is no slip or bond creep before cracking between concrete and reinforcement.

So far as the working load is within the allowable limit, most of these assumptions will be valid, however, it is considered important to approach the actual behavior of cracking and its effect on the stress distribution in reinforcement and to evaluate the significance of it in the cracking problem. In this study, therefore, the first approach was done by Elastic Theory or Modified Elastic Theory as usually done in reinforced concrete theories on these assumptions. When considered necessary, new assumptions are introduced in addition to these basic ones, which also can be easily justified. The second approach was the trials to find the influences of these fundamental assumptions on the stresses in concrete and reinforcement with or without computation.

2. Notation

ε_s =strain in steel

ε_c =strain in concrete

E_s =modulus of elasticity for steel

E_c =modulus of elasticity for concrete in compression
E_{ct} =modulus of elasticity for concrete in tension
E_{ct} =secant modulus of elasticity for concrete
E_{α} =sustained modulus of elasticity for concrete in tension
f_s =stress in steel
f_{s0} =stress in steel at the initial cracking in concrete
f_{sc} =stress in steel at cracked sections
f_c =stress in concrete in compression
f_t =stress in concrete in tension
f_{tu} =tensile strength for concrete
f'_c =compressive strength for concrete, by standard cylinder, at 28 days
f_y =yield point for steel
A_c =cross sectional area of concrete
A_s =cross sectional area of steel$=\dfrac{N\pi D^2}{4}$
A_t =area of concrete affected by the extension of steel
D =diameter of round bar having an area of $A_s/N=\dfrac{\sqrt{4A_s}}{\sqrt{N\pi}}$
N =number of reinforcement
p =ratio of cross sectional area of reinforcement to that of concrete
$U(x)$ =bond stress calculated as for a round bar
U_m =ultimate bond stress
W =average width of tensile cracks in concrete on the level of reinforcement
W_s =average width of cracks at surface of concrete
L =spacing of cracks in concrete
L_{min} =minimum spacing of cracks in concrete
P =summation of perimeters of bars in the area $A_c=N\pi D$
x =distance from a crack, measured along the reinforcement
F =axially applied load
M =applied bending moment which is considered constant along x
T =total tensile force in tensile reinforcement
C =total compressive force in concrete
C' =total compressive force in compressive reinforcement
n =ratio of the modulus of elasticity for steel to the secant modulus of elasticity for concrete
Δ_s =elongation of steel between the two adjoining cracks
Δ_c =elongation of concrete between the two adjoining cracks
ϕ =ratio of the assumed effective area to the fully developed area of

concrete, introduced in the concept of a *Hypothetical Cylindrical Area* in the flexural reinforced concrete member

m = factor determining the diameter of the concrete area affected by the extension of the reinforcing steel

h = total depth of beam subjected to pure bending

kd = depth of compressive concrete in the beam

L' = span length of beam subjected to pure bending

b = width of beam

d = depth of beam

p' = ratio of cross sectional area of compressive reinforcement to that of concrete

Part I. Cracking in Symmetrically Reinforced Members Subjected to the Axial Force

3. Stresses prior to the Initial Cracking

By Elastic Theory, $\varepsilon_s=\varepsilon_c$, i. e., $\frac{f_s}{E_s}=\frac{f_t}{E_{ct}}$. Therefore,

$$f_s=f_t\frac{E_s}{E_{ct}} \quad \text{(3-1)}$$

Although E_{ct} is a variable depending on the properties of concrete and stress condition, it may be considered as a constant which is equal to E_{ci}, if $f_t<f_{tu}$. Then

$$f_s=\frac{E_s}{E_{ci}}f_t \quad \text{(3-2)}$$

From the equilibrium,

$$f_s=\frac{nF}{A_c(1+np)} \quad \text{(3-3)}$$

At the initial cracking, Eqs. (3-1) and (3-2) become

$$f_{s0}=f_{tu}\frac{E_s}{E_{ct}} \quad \text{(3-4)}$$

$$f_{s0}=f_{tu}\frac{E_s}{E_{ci}} \quad \text{(3-5)}$$

That is, the steel stress in the initial cracking depends upon the tensile strength of concrete and modular ratio E_s/E_{ct} or E_s/E_{ci} and is independent on the dimension. After the initial cracking has occured, Eq. (3-3) is no more valid and stresses in reinforcement and in concrete will change their distribution.

In the above development, the effects of p and D are neglected which may affect to the strain distribution in concrete surrounding reinforcing bars through bond stresses. Therefore, there will be heterogeneous stress distribution in concrete and some discrepancy between the actual value and the computed one for f_{s0}. If the load is sustained, the value E_{ct} is no more constant and creep in concrete will permit gradual increase of f_{s0}. Shrinkage also affects adversely, producing compressive stress in reinforcement and tensile stress in concrete, of which magnitude depends on the conditions of restraint to free shrinkage.

4. Stresses just after Cracking

From the equilibrium between a cracked and an uncracked sections,

$$f_{sc}A_s = f_sA_s + f_tA_c \quad \text{......(4-1)}$$

$$f_t = \frac{P}{A_c}\int_0^x U(x)\,dx = \frac{D}{4p}\int_0^x U(x)\,dx \quad \text{......(4-2)}$$

Then the stress in the reinforcement will be

$$f_s = f_{sc} - \frac{4}{D}\int_0^x U(x)\,dx \quad \text{......(4-3)}$$

If the axial force F is increased, the tensile stress in concrete will reach its ultimate value and another crack will occure. From the symmetrical condition, the tensile stress in concrete between the two adjoining cracks may reach its maximum value at $x=L/2$, and simultaneously the tensile stress in reinforcement will be at the minimum value.

$$(f_t)_{x=L/2} = \frac{4p}{D}\int_0^{L/2} U(x)\,dx \quad \text{......(4-4)}$$

$$(f_s)_{x=L/2} = f_{sc} - \frac{4}{D}\int_0^{L/2} U(x)\,dx \quad \text{......(4-5)}$$

From the above discussion, it is obvious that the stress distributions in concrete and reinforcement are dependent upon that of bond stress. Therefore, if the value of $U(x)$ is known, the values of f_t or f_s will be exactly estimated. Unfortunately there is not much available data on the bond stress distribution. According to Watstein and Parsons (Ref. 15), $U(x)$ can be expressed as following:

$$U(x) = U_m \quad \text{......(1')}$$

$$U(x) = U_m\left(1 - \frac{2x}{L}\right) \quad \text{......(2')}$$

$$U(x) = U_m\left(1 - \frac{4x^2}{L^2}\right) \quad \text{......(3')}$$

Also Bertero suggests $U(x)$ may be expressed as one of the harmonic functions of x, (Ref. 1), of which the most simple case will be as following:

$$U(x) = U_m \sin\left(\frac{2\pi x}{L}\right) \quad \text{......(4')}$$

Physically Eq. (1') is equivalent to the uniform stress distribution, which seems almost impossible in actual conditions, but is sometimes used by designers because of its simplicity. Although Eqs. (2') and (3') do not satisfy the boundary condition at the cracked section, where the bond stress must be equal to zero, they yield the comparable values as Eq. (4') does. Replacing $U(x)$ by these relations,

$$\int_0^{L/2} U_m dx = \frac{L}{2} U_m \quad \cdots\cdots (1'')$$

$$\int_0^{L/2} U_m\left(1-\frac{2x}{L}\right) dx = \frac{L}{4} U_m \quad \cdots\cdots (2'')$$

$$\int_0^{L/2} U_m\left(1-\frac{2\pi x}{L^2}\right) dx = \frac{L}{3} U_m \quad \cdots\cdots (3'')$$

$$\int_0^{L/2} U_m \sin\left(\frac{2\pi x}{L}\right) dx = \frac{L}{\pi} U_m \quad \cdots\cdots (4'')$$

Taking the ratio among these four values,

$$(1'') : (2'') : (3'') : (4'') = 1 : 1/2 : 2/3 : 2/\pi$$
$$= 1 : 0.5 : 0.67 : 0.64$$

As is obvious in the preceding computation, there is no essential difference among the last three values. Considering the validity of the basic assumptions, any of these three Equations, (2'), (3'), and (4') can be used for practical purposes. The actual bond-stree distribution may be somewhat like illustrated in Section 6.

Assuming $U(x) = U_m \sin\left(\frac{2\pi x}{L}\right)$,

$$\int_0^{x} U_m \sin\left(\frac{2\pi x}{L}\right) dx = \frac{L}{2\pi} U_m\left(1-\cos\frac{2\pi x}{L}\right)$$

$$f_t = \frac{4p}{D} \cdot \frac{L}{2\pi} U_m\left(1-\cos\frac{2\pi x}{L}\right) \quad \cdots\cdots (4\text{-}6)$$

$$f_s = f_{sc} - \frac{4L}{2\pi D}\left(1-\cos\frac{2\pi x}{L}\right) \quad \cdots\cdots (4\text{-}7)$$

for $x = L/2$,

$$f_t = \frac{4pL}{\pi D} U_m \quad \cdots\cdots (5')$$

$$f_s = f_{sc} - \frac{4L}{\pi D} \quad \cdots\cdots (6')$$

Expanding Eqs. (5') and (6') to the general expressions,

$$(f_t)_{x=L/2} = K_2 \frac{4p}{D} \frac{L}{2} U_m \quad \cdots\cdots (4\text{-}8)$$

$$(f_s)_{x=L/2} = f_{sc} - K_2 \frac{4}{D} \frac{L}{2} U_m \quad \cdots\cdots (4\text{-}9)$$

where K_2 = a coefficient depending upon the bond stress distribution.
Substituting Eq. (4-8) into Eq. (4-9),

$$(f_s)_{x=L/2}=f_{sc}-\frac{1}{p}(f_t)_{x=L/2} \quad \cdots\cdots (4\text{-}10)$$

5. Minimum Spacing and Width of Cracks

With further increase of f_s, a new crack may form between the two adjoining cracks, and repeating this process the spacing of the cracks becomes smaller and smaller until a limiting value of spacing is reached, at which the tensile stress in concrete does not exceed the tensile strength. At the center of the two adjoining cracks in the minimum spacing, the stress in concrete can be assumed to be its ultimate value, therefore, the following expressions can be derived.

From Eq. (4-8),

$$L_{min}=\frac{2}{K_2}\frac{D}{4p}\frac{f_{tu}}{U_m} \quad \cdots\cdots (5\text{-}1)$$

From Eq. (4-10),

$$(f_s)_{L_{min}}=f_{sc}-\frac{1}{p}f_{tu} \quad \cdots\cdots (5\text{-}2)$$

If there is no slip between concrete and steel, as assumed,

$$\begin{aligned}
W&=\text{steel elongation—concrete elongation}\\
&=\Delta_s-\Delta_c\\
&=2\int_0^{L/2}\frac{f_s(x)}{E_s}dx-2\int_0^{L/2}\frac{f_t(x)}{E_{ct}}dx\\
&=\frac{2}{E_s}\int_0^{L/2}\left(f_{sc}-\frac{4}{D}\int_0^{L/2}U(x)dx\right)dx\\
&\quad-\frac{2}{E_{ct}}\int_0^{L/2}\frac{4p}{D}\left(\int_0^{L/2}U(x)dx\right)dx \quad \cdots\cdots (5\text{-}3)
\end{aligned}$$

where E_{ct} is assumed as a constant. From the discussion in the previous section,

$$\int_0^{L/2}U(x)dx=k_2U_m$$

Therefore,

$$\begin{aligned}
W&=\frac{2}{E_s}\int_0^{L/2}\left(f_{sc}-\frac{4}{D}k_2U_m\right)dx-\frac{2}{E_{ct}}\int_0^{L/2}\frac{4p}{D}k_2U_mdx\\
&=\frac{L}{E_s}\left(f_{sc}-\frac{4}{D}k_2U_m\right)-\frac{L}{E_{ct}}\frac{4p}{D}k_2U_m \quad \cdots\cdots (5\text{-}4)
\end{aligned}$$

If we take the ratio of Δ_c to Δ_s,

$$\Delta_c/\Delta_s=\left(E_sk_2U_m\frac{4p}{D}\right)\Big/E_{ct}\left(f_{sc}-k_2U_m\frac{4p}{D}\right) \quad \cdots\cdots (5\text{-}5)$$

Assuming $f_{sc}=f_y=2{,}300\ \mathrm{kg/cm^2}$ (33,000 psi)

$D=1.27$ cm (0.5 in.),

$E_s=2.1\times10^6\ \mathrm{kg/cm^2}$ (30×10^6 psi)

$p=0.04$

$k_2=2/\pi=0.64$

$U_m=21\ \mathrm{kg/cm^2}$ (300 psi)

And substituting these values into Eq. (5-5), $\Delta_c/\Delta_s=0.2$. Although this computation is based on the particular values, it suggests that the magnitude of Δ_c is very small in comparison with that of Δ_s. Moreover, considering the effect of shrinkage which reduces the strain in concrete and that of creep which increases the steel stress considerably, Δ_c can be appropriately neglected. If there is any bond-creep against the assumption, to neglect Δ_c will be more reasonable, because the bond-creep increases the crack width W directly by the amount of slip and decreases the strain in concrete.

Then Eq. (5-4) becomes

$$W=\frac{L}{E_s}\left(f_{sc}-\frac{4}{D}k_2U_m\right) \qquad \text{(5-6)}$$

At the minimum spacing of cracks,

$$W=\frac{1}{E_s}\left(f_{sc}-\frac{1}{p}f_{tu}\right)L_{min} \qquad \text{(5-7)}$$

If the effect of concrete elongation is considered,

$$W=\frac{1}{E_s}\left(f_{sc}-\frac{1}{p}f_{tu}\right)L_{min}-\frac{1}{E_s}f_{s0}L_{min} \qquad \text{(5-8)}$$

$$=\frac{1}{E_s}(f_{sc}-f_{s0}-f_{tu}/p)L_{min} \qquad \text{(5-8')}$$

Eq. (5-8') coincides with what is suggested by Bertero and others. (Ref. 1 and 4). In Eq. (5-7), when f_{sc} approaches to the yield point, the second term becomes smaller comparing with the first, and to neglect the term will give a conservative value for W, which is given by

$$W=\frac{f_{sc}}{E_s}L_{min} \qquad \text{(5-9)}$$

This formula is the same often refered by Wästlund and others. (Ref. 11).

Rewriting Eq. (5-8),

$$W=\frac{1}{2k_2E_s}\left(f_{sc}-f_{s0}-\frac{f_{tu}}{p}\right)\frac{D}{p}\frac{f_{tu}}{U_m} \qquad \text{(5-10)}$$

Introducing a new coefficient k,

$$k=1-\frac{1}{f_{sc}}\left(f_{s0}+\frac{f_{tu}}{p}\right)$$

$$W=\frac{k}{E_s}f_{sc}\cdot\frac{1}{2k_2}\frac{D}{p}\frac{f_{tu}}{U_m} \qquad \text{(5-11)}$$

6. Illustrative Presentation of Srress Distriburion

The relations obtained in the preceeding analysis and discussion in Sections 3, 4, and 5 are illustrated in this Section. Fig. 1 illustrates the stress distribution prior to the initial cracking. In Fig. 2, the stress distribution immediately after the initial cracking has occured and in Fig. 3, the one after the following cracks have developed are presented. Figs. 4 through 8 present the assumed bond-stress distributions and their corresponding longitudinal strees distributions in concrete and reinforcement at the minimum spacing of cracks. Finally in Fig. 9, the one after the minimum spacing of cracks has been reached is illustrated.

7. Influence of Shear Deformation

In the development of the discussion, the effects of shearing deformation in the concrete were disregarded. Because of this deformation, the tensile stress in the concrete is at its maximum in the vicinity of the reinforcement and decreases with the distance from the reinforcement. The stress in the concrete is not always positive, i. e., it can be in compression. Accordingly the resistivity of concrete to tension must be less than that indicated by the equations, in effect this is equivalent to the reduction of the cross sectional area A_c or an increase in p. Therefore, the effects of shearing deformation would be somewhat related with the diameter and the cross sectional area of reinforcement.

8. Influence of Slip between Concrete and Reinforcement

Slip in bond, or creep in bond under sustained load releases the strain in concrete, consequently increasing stress in reinforcement and decreasing stress in concrete. Jnst prior to slipping, the bond stress reaches to the ultimate value and suddenly decreases to zero with the slipping and the slipped depth is directly transformed into the width of cracks. Accordingly the width of cracks may be expressed as following:

$$W=\Delta_s-\Delta_c+\text{Slipped Depth}$$

According to Watstein and Parsons (Ref. 15), however, the effect of slip is so small in comparison with other movements, Δ_s and Δ_c, that it will be appropriately neglected except when the tensile stress in concrete is large at early ages and when members reinforced with plain-bars are subjected to repeated loading.

9. Loading beyond the Proportional Limit of Steel

After the minimum spacing of cracks has reached, the further increase in load F expands cracks without any increase in their numbers until the stress in reinforcement reaches to the yield point at cracked sections. The formulae (5-10) and (5-11) are no more valid because the depth of slip will also increase. Under this situation, the application of Eq. (5-9) will be proper to estimate the crack width. If the reirforcement are deformed bars, the separating effects and the shearing effects will play important rolls due to the act of projected parts. Crushing of concrete along the reinforcement and spalling of concrete in the vicinity of the cracked sections will be observed. After f_{sc} has reached to the yielding point, crack widths will increase without adding any load, the depth of slip will be also deepened more and more, producing the redistribution of stresses in the reinforcement and concrete. Finally the stresses in reinforcement will become roughly uniform as illustrated in Fig. 9.

10, Summary

Reviewing the works done by Bornenman, Colonetti, Saliger, Watstein, and others, (Ref. 3, 6, 10, 14, 15, and 16), they are summarized as following:

(1) the width of cracks is roughly proportional to D/p,

(2) the rate of increase of the width is independent of the strength of concrete, and

(3) the stress at the initial cracking is roughly proportional to the strength of concrete.

These findings present the basic foundations for the analysis developed in the preceeding sections. From the analysis and discussion, it will be easily found that the reinforcement provided with a mechanical bond system, such as deformed bars, is more effective than smooth bars not only in controlling the initial width of cracks, bnt also in minimizing the enlargement of cracks caused by sustained or repeated loads. It is also obvious that the use of high-tensile-strength steel is of great advantage, of which adverse effects to increase the crack-width can be easily canceled by the proper methods.

Part II. Cracking in Single-Reinforced Members Subjected to Pure Bending

11. Prior to the Initial Cracking

By Elastic Theory, (refer to Fig. 11),

$$\varepsilon'_c = \frac{h-kd}{d-kd}\varepsilon_s = \frac{h-kd}{d-kd}\,\frac{f_s}{E_s} = \frac{f_t}{E_{ct}}$$

$$\therefore f_s = \frac{E_s}{E_{ct}}\,\frac{d-kd}{h-kd} f_t \qquad \text{(11-1)}$$

Just prior to the intial cracking, f_t will be equal to f_{tu}, then,

$$f_{s0}=\frac{E_s}{E_{ct}}\frac{d-kd}{h-kd}f_{tu} \qquad (11\text{-}2)$$

where,

$$kd=\frac{bh^2+2ndA_s}{2(bh+(n-1)A_s)} \qquad (11\text{-}3)$$

Comparing E_q (11-2) with Eq. (3-4), if $h \doteqdot d$, then the former will be reduced to the latter. This approximation is considered very important in introducing the Hypothetical Cylindrical Area by Chi and Kirstein (Ref. 4), which will be discussed in Section 13. The bending moment for the initial cracking will be given by

$$M_0=\frac{bd}{3}\left(\frac{h}{d}(h-kd)+d^2(3-k)\frac{(1-k)}{(h-kd)}np\right)f_{tu} \qquad (11\text{-}4)$$

12. Just after the Initial Cracking has Occured

At the cracked section (Fig. 12), by Modified Elastic Theory or Straight Line Theory,

$$k=\frac{-2nA_s+\sqrt{4n^2A_s^2+8nbdA_s}}{2bd}$$

$$=\sqrt{n^2p^2+2np}-np \qquad (12\text{-}1)$$

$$jd=d-\frac{1}{3}kd=\frac{d}{3}(3+np-\sqrt{n^2p^2+2np})A_s \qquad (12\text{-}2)$$

From the equilibrium,

$$f_{sc}=\frac{M_0}{jdA_s}=\frac{3M_0}{d(3+np-\sqrt{n^2p^2+2np})A_s} \qquad (12\text{-}3)$$

Replacing M_0 by Eq. (11-4),

$$f_{sc}=\frac{b\left(\frac{h}{d}(h-kd)+\frac{d^2(3-k)(1-k)}{h-kd}np\right)f_{tu}}{(3+np-\sqrt{n^2p^2+2np})A_s}$$

$$=\frac{h(h-kd)^2+d^3(3-k)(1-k)np}{d^2(h-kd)(3+np-\sqrt{n^2p^2+2np})p}f_{tu} \qquad (12\text{-}4)$$

where kd is given by Eq. (11-3).

That is, before the initial cracking takes place, the stress in reinforcement distributes uniformly over the span L', where the pure bending moment is acting. Although the magnitude of the stress in concrete varies, it is also uniform on the same level of fiber. Just after the initial cracking, stresses are redistributed producing the maximum stress in reinforcement at the cracked section. The stresses at uncracked sections sufficiently far from the cracked section will be approximately given by Eq. (11-1), where f_t will be solved by the Elastic Theory using the transformed section. Therefore, the stress distribution is somewhat similar to that shown in Fig. 2.

13. After the following Cracks have developed (1)

To attain simplicity and reasonable validity of the equations, Chi and Kirstein (Ref. 4) have developed a semi empirical approach to the flexural cracking problem by introducing a Hypothetical Cylindrical Area in the tensile zone of the beam subjected to bending. Their analysis was besed on the following assumptions:

(1) Concrete is a homogeneous and elastic material.

(2) Reinforced steel is not stressed past its proportional limit.

(3) The cracked portion of the beam is subjected to pure bending.

(4) After a number of cracks have occured in the tensile zone of the concrete, the tensile strains due to flexure are neglesible. Any measurable strains in that portion of the concrete are attributed to the shear deformation developed through bond by the extension of the steel.

The assumptions (1) and (2) are same as made in section I, and (3) is to simplify the analysis as objected in this discussion, which will be mentioned later. If the assumption (4) is valid, the tensile strains in concrete can be considered as caused by pure tension, although the actual distribution of stress is not symmetrical. So far as the Straight Line Theory is reserved, that is, plane sections remaine plane, the stresses in concrete in the tensile zone of the member have to be proportional to the strains at uncracked sections. The stresses in the compressive zone are not necessarily proportional and proper stress blocks can be assumed according to the external moment. The neutral axis is no more on the same level, and the stress in concrete in any uncracked section will be somewhat like shown in Fig. 14. According to the assumption (4), the fiber stress is at its maximum on the level of reinforcement which contradict to the Straight Line Theory, on which the hypothetical theory is based. Therefore, the assumption (4) seems doubtful, on which Bertero also has expressed his misgiving (Ref. 2). Considering the concrete stress after a number of cracks have been developed, however, it will be close to its ultimate value at the center of the two adjoining cracks. If the cover thickness d' is very small comparing to the depth d, h ≒ d can be reasonably assumed, which leads to the approximation mentioned in Section 11, i. e.,

$$f_{s0}=\frac{E_s(d-kd)f_{tu}}{E_{ct}(h-kd)}=\frac{E_s f_{tu}}{E_{ct}} \qquad \text{(13-1)}$$

Although the concrete stress on the level of reinforcement is greater than that on the surface of tension side according to Chi and Kirstein, the difference between them will be small and can be neglected. Then the steel stress will be approximately equal to that given by Eq. (13-1).

So far as the conventionally designed members are concerned except thin slabs, the above discussion will be valid, although it was developed on the successive approximation and assumption. This may be the reason why the test results are in good agreement with the semi empirical theory presented by Chi and Kirstein in spite of the contradicted assumption. It also signifies that the concept of Hypothetical Cylindrical Area can be applied without erroneous results for the computation of stresses at the uncracked sections. Using the concept, Eq. (4-4) is reduced to Eq. (13-2),

$$(f_t)_{x=L/2}=\frac{4p}{D}\int_0^{L/2}U(x)dx \qquad (4\text{-}4)$$

If a linear stress distribution is assumed for the bond,

$$\int_0^{L/2}U(x)dx=\int_0^{L/2}U_m\left(1-\frac{2x}{L}\right)dx=U_m\frac{L}{4}$$

For $L=L_{min}$ and $f_{tu}=\frac{p}{D}U_mL_{min}$, and replacing p by p_e,

$$L_{min}=\frac{D}{p_e}\frac{f_{tu}}{U_m}$$

where $p_e=\frac{A_s}{\phi A_t}=\frac{A_s}{\phi A_s m^2}=\frac{1}{m^2\phi}$, therefore,

$$L_{min}=m^2\phi D\frac{f_{tu}}{U_m} \qquad (13\text{-}2)$$

The formulae for the average crack width at the minimum spacing will be obtained from Eq. (5-8).

$$W=\frac{1}{E_s}\left(f_{sc}-f_{s0}-\frac{f_{tu}}{p}\right)L_{min} \qquad (5\text{-}8)$$

Substituting $p=p_e$ into Eq. (5-8),

$$W=\frac{1}{E_s}(f_{sc}-f_{s0}-m^2\phi f_{tu})L_{min} \qquad (13\text{-}3)$$

If the concrete strain is neglected, Eq. (13-3) is reduced to

$$W=\frac{1}{E_s}(f_{sc}-f_{s0})m^2\phi D\frac{f_{tu}}{U_m} \qquad (13\text{-}4)$$

Formulae (13-2) and (13-4) agree with those suggested by Chi and Kirstein for the minimum crack spacing and the crack width, respectively. Moreover they suggest according to their empirical data and its analysis (Ref. 3 and 4),

$$m=4,\quad \frac{f_{tu}}{U_m}=\frac{5}{16},\quad \text{and}\quad f_{t0}=\frac{2500}{\phi D}\text{psi}$$

Using these values, Eqs. (13-2) and (13-4) are reduced to

$$L_{min}=5\phi D\ \text{in.} \qquad (13\text{-}5)$$

$$W=5\phi D\left(\frac{f_{sc}-\frac{2500}{\phi D}}{E_s}\right)\text{in.} \qquad (13\text{-}6)$$

Accordingly the average crack width on the concrete surface will be estimated by using the following formula.

$$W_s = 5\phi D \frac{h-kd}{d-kd}\left(\frac{f_{sc}-\dfrac{2500}{\phi D}}{E_s}\right) \text{ in.} \qquad (13\text{-}7)$$

where $k=(\sqrt{n^2p^2+2np}-np)$

$$f_{sc}=\frac{3M}{pbd^2(3+np-\sqrt{n^2p^2+2np}\,)}$$

On the application of Eqs. (13-5), (13-6), and (13-7), consideration have to be made to the fact that the ratio of tensile strength to bond strength depends on the deformation pattern and the surface roughness of reinforcing bars, therefore, the new evaluation of the ratio on the particular type of bars will be necessary for E_s and U_m, as well as for E_{ct} and f_{tu} on the concrete.

Wästlund and others suggest to employ the following formulae for crack-spacing and width (Ref. 11, 12, and 13).

$$c_1 = L_{min} + c_2\frac{f_{tu}A_c}{U_m n\pi D} = c_1 + c_2\frac{f_{tu}D}{4p} \qquad (13\text{-}8)$$

where c_1 ann c_2 are cofficients.

$$W = L_{min}\frac{f_{sc}}{E_s} \qquad (13\text{-}9)$$

For pure bending, they also suggest,

$$W_{max} = c_3 D\left(\frac{I_c f_s}{DA_s E_s (h-kd)}\right)^{2/3} \qquad (13\text{-}10)$$

where c_3=coefficient depending on the type of bars, 0.23 for plain bars, 0.16 for deformed bars,

I_c=moment of inertia of the full concrete section about the neutral axis, and

f_s=working stress in reinforcement.

14. After the Following Cracks have Developed (2)

Reviewing the procedures through which the formulae shown in the previous sections have been derived, it does not seem proper to apply them directly for evaluation of stresses in reinforcement or concrete, because these formulae contain successive approximations and some of them include coefficients determined empirically. However, the concept of Hypothetical Cylindrical Area can be used advantageously for stress analysis at uncracked sections.

For a given value of bending moment M, the stresses at uncracked and cracked sections will be found by Elastic and Straight Line Theories, respectively, as shown in Sections 11. and 12.

$$f_{s0}=\frac{E_s}{E_{ct}}\,\frac{d-kd}{h-kh}f_{tu} \quad \text{(11-2)}$$

$$f_{sc}=\frac{3M}{pbd^2(3+np-\sqrt{n^2p^2+2np})} \quad \text{(12-3)}$$

Between these two different values, there must be continuous transitive stress distribution which is developed through bond stress. Moreover, from the symmetry, steel stress may be at its minimum and contrarily concrete stress at its maximum at the center of the two adjacent cracked sections. Considering a portion of tensile zone in a flexural member as an axially loaded prism, in other words, applying the concept of Hypothetical Cylindrical Area, the required stress distribution may be obtained without so many errors.

In axially loaded members,

$$f_t=\frac{4p}{D}\int_0^x U(x)\,dx \quad \text{(4-2)}$$

$$f_s=f_{sc}-\frac{4}{D}\int_0^x U(x)\,dx \quad \text{(4-3)}$$

Replacing p by $p_e=1/(\phi m^2)$

$$f_t=\frac{4}{Dm^2\phi}\int_0^x U(x)\,dx \quad \text{(14-1)}$$

At the center between two adjoining cracks,

$$f_t=\frac{4}{Dm^2\phi}\int_0^{L/2} U(x)\,dx=\frac{4}{Dm^2\phi}k_2U_m\frac{L}{2} \quad \text{(14-2)}$$

$$f_s=f_{sc}-\frac{4}{D}k_2U_m\frac{L}{2} \quad \text{(14-3)}$$

In this case also, the stress distribution is determined by the bond stress. As discussed in Part I, any of linear, parabolic, or harmonic bond stress distribution may be assumed. If a linear distribution is assumed, at $L=L_{min}$, on the level of reinforcement,

$$f_t=\frac{1}{Dm^2\phi}U_mL_{min}\frac{d-kd}{h-kd}f_{tu}$$

Therefore,

$$L_{min}=m^2\phi D\cdot\frac{h-kd}{d-kd}\cdot\frac{f_{tu}}{U_m} \quad \text{(14-4)}$$

If the thickness of cover for reinforcement is neglesible, $(h-kd)/(d-kd)$ becomes equal to one, Eq. (14-4) is reduced to Eq. (13-2). Stress in reinforcement is not affected by the values of m and ϕ. Accordingly the stress distribution for bond, concrete, and reinforcement will be such as illustrated in Fig. 3 through 9, where f_{sc} has to be replaced by the value given in Eq. (14-1) and also f_{tu}, f_t, and p have to be rewritten by $(d-kd)f_{tu}/(h-kd)$, $4U_m(x)f(x)/(m^2\phi D)$, and p_e, respectively. The function $f(x)$ is determined by the pattern of the bond stress distribution.

Since the bending moment is constant along the span L', at the center of the uncracked section,

$$M=pbd^2(1-k)f_s+\frac{bh(h-kd)}{3}f_{tu}\quad\cdots\cdots(14\text{-}5)$$

at the cracked section,

$$M=\frac{1}{3}pbd^2(3+np-\sqrt{n^2p^2+2np})f_{sc}\quad\cdots\cdots(14\text{-}6)$$

From Eqs. (14-5) and (14-6),

$$(f_s)_{x=L/2}=\frac{1}{3(1-k)}(3+np-\sqrt{n^2p^2+2np})f_{sc}-\frac{h(h-kd)f_{tu}}{3pd^2(1-k)}\quad\cdots\cdots(14\text{-}7)$$

The stress distribution at this stage is illustrated in Fig. 13.

15. Validity of the Analysis

At the initial stage of loading, i.e., for small magnitude of bending moment, concrete in the uncracked section will resist the flexural tensile stress and the assumption of concrete being elastic and homogeneous material will be reasonably justified. When the tensile strength of concrete in flexure is overcome, the initial crack will appear with irregularity along the span. Further increase of bending moment will develop cracks closer until the minimum spacing is reached. However, under normal working stress, $f_c<0.45f_c'$, width of cracks are small and the stress distribution in compressive zone will be practically linear. Although the neutral axis moves towards compressive zone, accordingly redistribution of stresses takes place, the Straight Line Theory will remain valid up to this stage. When the bending moment increases, producing stress $f_c>0.45f_c'$, the stress in compressive zone is no more linear, and the crack height and width will increase. If steel stress is below yielding point, the stress may be estimated with certain accuracy on the assumption that the plane section remains plane, and applying appropriate stress blocks for oompressive zone. The relations derived in the previous sections are no more valid upon this stage. At this stage, creep of bond is anticipated no more neglesible and it will increase the crack width and stresses in reinforcement as discussed in Section 8.

The further increase in bending moment beyond this stage will produce the three types of behaviors, according to the properties of the section, i.e., under-reinforced, balanced, or over-reinforced section. (1). If under-reinforced, the yielding of steel will start and finally the member will fail by crushing of the compressive zone of concrete. Before the failure, an excessive curvature will be also observed. In this case slip between concrete and reinforcement will be not so large as observed in the case of axial cracking, because a certain amount of friction will be developed between concrete and reinforcement due to the excessive curvature which resists to the slip. Even so, the slip depth will be enlarged and the spalling of concrete may be observed near the cracked sections if the member is reinforced with deformed bars. Through this process, nearly uniform distribution of stress in reinforcement is anticipated prior to the final failure. (2). If over-reinforced, the member will fail by sudden crushing of compression concrete without yielding of steel. (3). If balanced, the crushing of concrete and the initial yielding of steel will take place simultaneously and fail. In the cases of (2) and (3), nearly exact behavior can be forecast up to the stage of failure through computations, although it is not so significant but in a case of experimental study.

16. Effect of Shear

In the preceeding analysis, the effects of shear was disregarded, i.e., the constant bending moment along the span lehgth was assumed for analysis. When the bending moment is not constant, however, its effects must be considered: especially when the maximum bending moment and shear coexist, as in a case of continuous beam or fixed end beam, it will be very important. Since it is beyond the subject of this study, however, the method of approach to this problem is briefly stated below.

Prior to the formation of cracks, reinforced concrete beams have stresses quite similar to those of a homogeneous beam, even under the combined action of longitudinal tension originated by bending and transverse shearing forces. At this stage, in other words, shear does not contribute to an appreciable extent in the stress distribution. When a crack have formed shearing and diagonal tensile stress cannot be transmitted accross the opening by the concrete or reinforcement alone, i.e., the stress redistribution will take place both in concrete and in reinforcement. However, most of beams have web reinforcement which is also subjected to the stress redistribution, and the mode of it varies considerably depending upon the type of web reinforcement, such as no web reinforcement, vertical stirrups, bent-up bars, bent-up bars and vertical stirrups, or orthogonal web reinforcement. Therefore, conventional approach has been made for each type of web reinforcements on the following assumptions in addition to those stated in Section 1,. although some of them are doubtful in their validity,

(1) Concrete resist part of total shear.

(2) Web reinforcement resist the balence of the total shear.

(3) All stirrups resist the same force.
(4) Idealized crack is inclined to beam axis at angle 45°.
(5) Cracking extends up to the neutral surface of the beam.

17. Significane of Cracking Problem

In the design of reinforced concrete members, cracking will be of little importance, if there is no danger of corrosion and anticipated loads are always static and estimable. Generally speaking, however, the corrosion of reinforcement will be of primary importance, which decides span of life of structures providing of their being properly designed for conceivable stress conditions, although the corrosive effects depend upon the environmeotal factors. If structure is a maritime structure located in or near the sea water, or exposed to the moist severe conditions, the crack-spacing and width are as important as stress analysis. So for as cracking in concrete is not avoidable, the study is considered necessary on the relation between the corrosive effects and crack-width. In relation with this subject, the study on stress corrosion which may induce serious corrosion is also suggested.

Even when the corrosion is not an important factor, cracking may give rise to a serious problem. From the preceeding analysis, it is found that there is a difinite difierence of stresses between cracked and uncracked sections. This signifies that there is a sudden redistribuion of stresses in reinforcement upon the formation of cracks, no matter they are flexural or tensile cracks. It is obvious, therefore, that there may be stress concentration or local failure in reinforcing steel due to impact of loads or repeated loads, which may induce the structural failure. The crack formation in the vicinity of supports may imply the failure of anchorage if the sound anchorage is not provided.

18. Conclusion

(1) Although some modification is necessary for computation of the effects of cracks upon the stress distribution, there is no essential difference between pure tensile cracks and pure flexural cracks.
(2) Under the normal working stresses, the stress distribution in concrete or reinforcement depends on the stress distribution of bond.
(3) Bond stress distribution may be assumed as any of linear, parabolic, or harmonic distribution along reinforcement. Although there is an obvious difference on the boundary conditions, it does not affect practically to the stress distribution in concrete or in reinforcement. For practical purposes, therefore, the linear distribution may be preferably applied because of its simplicity.

(4) Concept of Hypothetical Cylindrical Area by Chi and Kirstein will be a useful tool for analysis of stresses in concrete in the tensile zone of the flexural members.

ACKNOWLEDGEMENTS

The author acknowledges his indebtedness to Professor Vitelmo V. Bertero, University of California, U.S.A., for his remarkable guidance and suggestions on carrying out this study.

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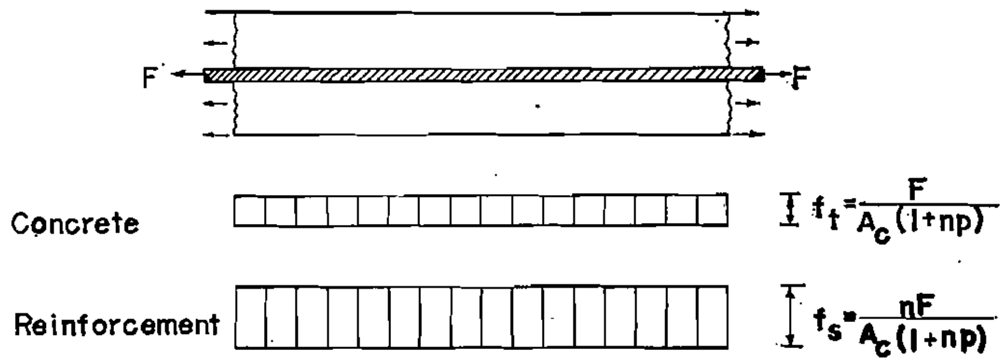
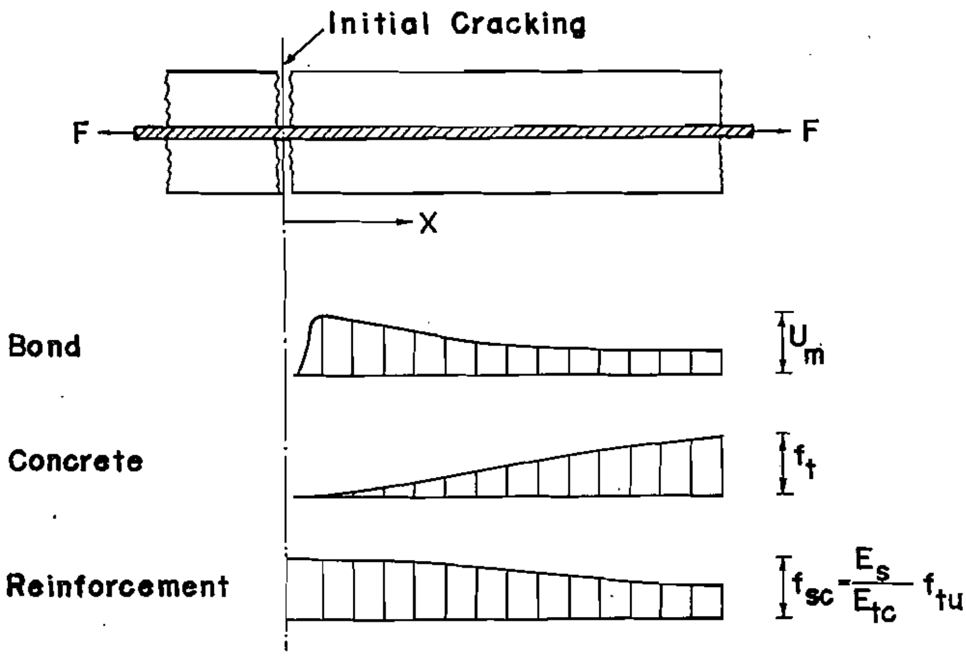
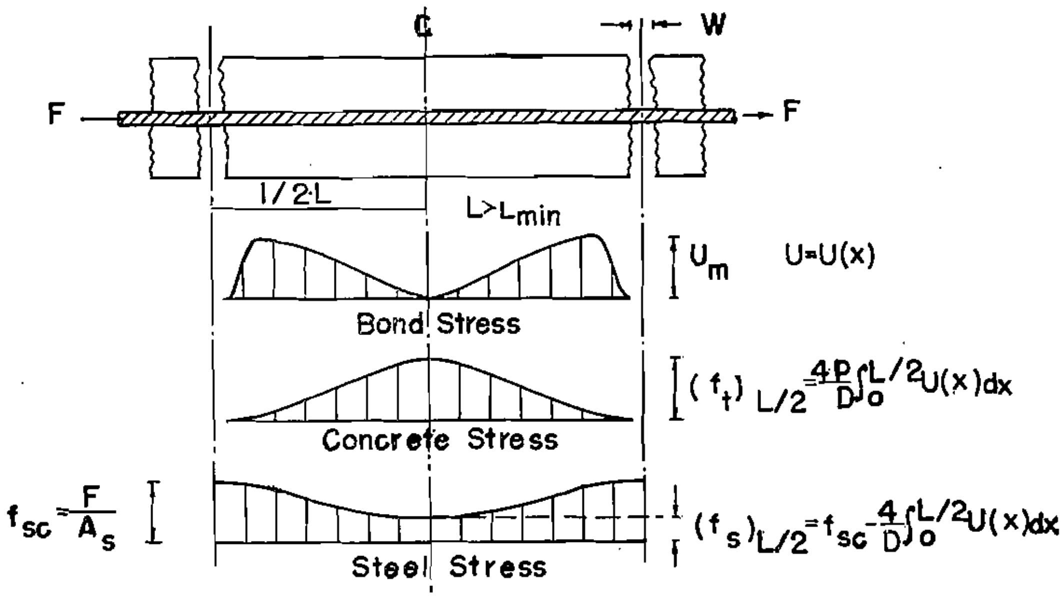


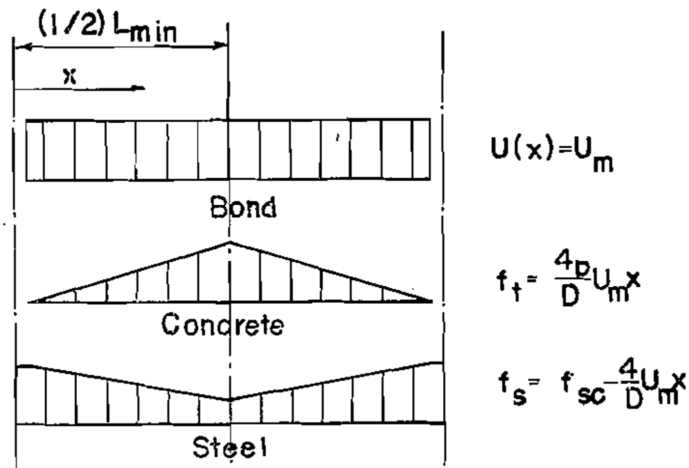
FIG·1, LONGITUDINAL STRESS DISTRIBUTION PRIOR TO THE FORMAION OF THE INITIAL CRACKING IN AXIALLY LOADED MEMBERS.



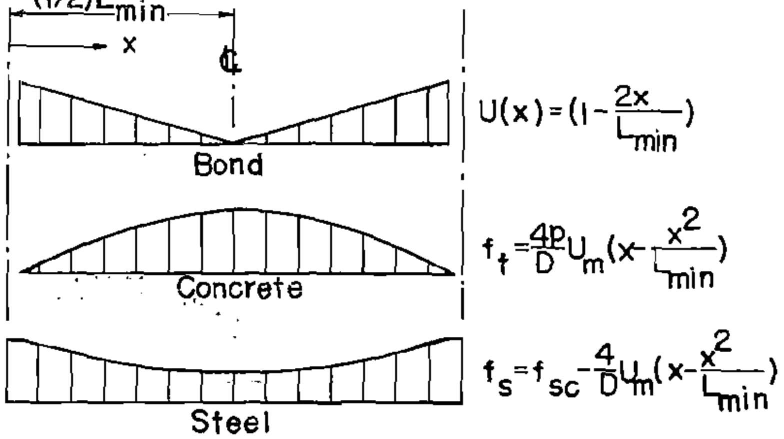
FIG·2, LONGITUDIAL STRESS DISTRIBUTION AFTER THE FORMATION OF THE INITIAL CRACKING IN AXIALLY LOADED MEMBERS.



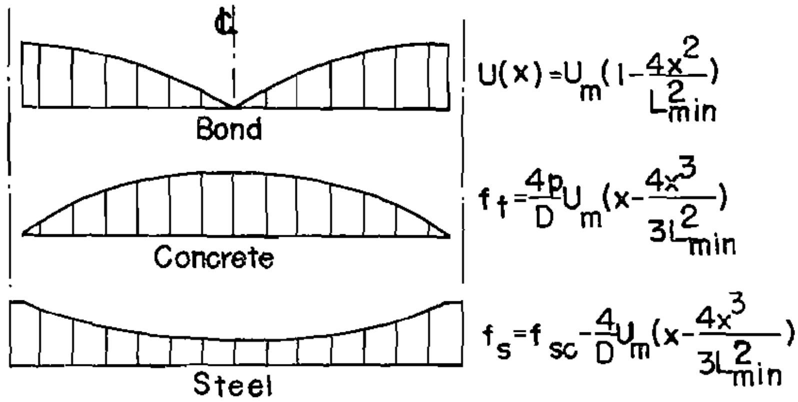
FIG·3. LONGITUDINAL STRESS DISTRIBUTION AFTER FOLLWING CRACKS HAVE DEVELOPED IN AXIALLY LOADED MEMBERS.



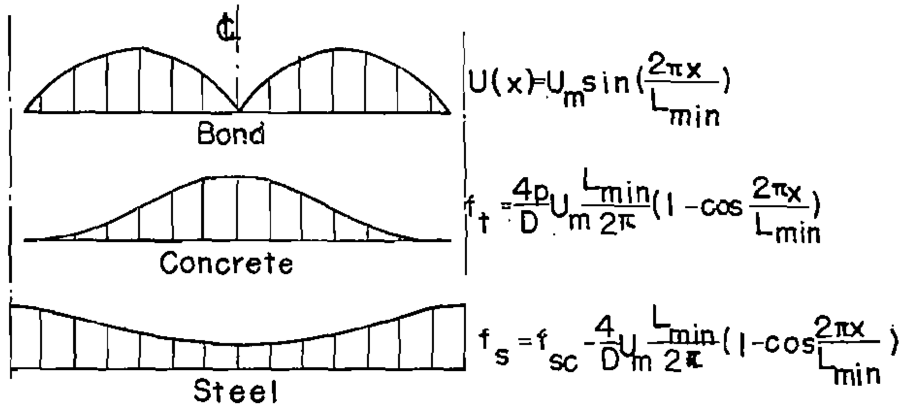
FIG·4. ASSUMED LONGITUDINAL STRESS DISTRIBUTION AT THE MINIMUM CRACK SPACING. BOND STRESS IS CONSTANT.



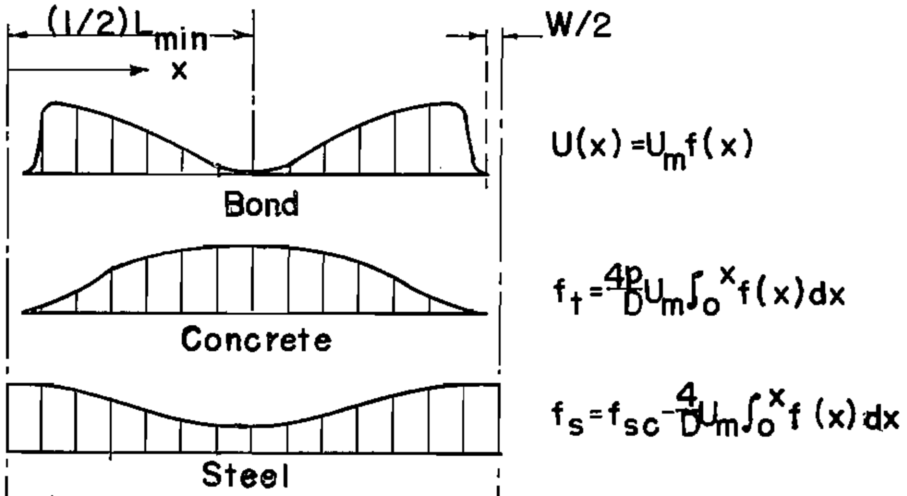
FIG·5, ASSUMED STRESS DISTRIBUTION AT THE MINIMUM CRACK SPCING. BOND STRESS IS LINEAR.



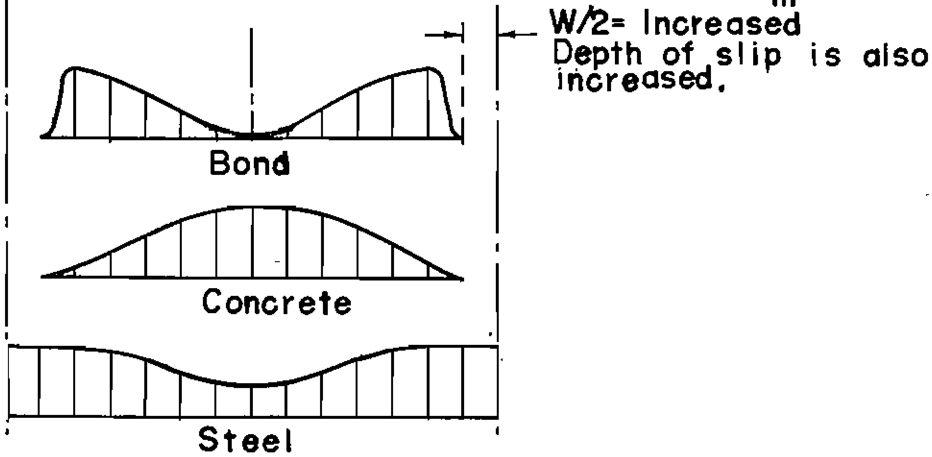
FIG·6, ASSUMED STRESS DISTRIBUTION AT THE MINIMUM CRACK SPCING. BOND STRSS IS PARABOLIC.



FIG·7, ASSUMED STRESS DISTRIBUTION AT THE MINIMUM CRACK SPACING. BOND STRESS HARMONIC.



FIG·8, ASSUMED STRESS DISTRIBUTION AT THE MINIMUM CRACK SPACING. FUNCTION OF BOND STRESS $U(x)=U_m f(x)$



FIG·9, STRESS DISTRIBUTION AFTER THE MINIMUN CRACK SPACING HAS DEVELOPED.

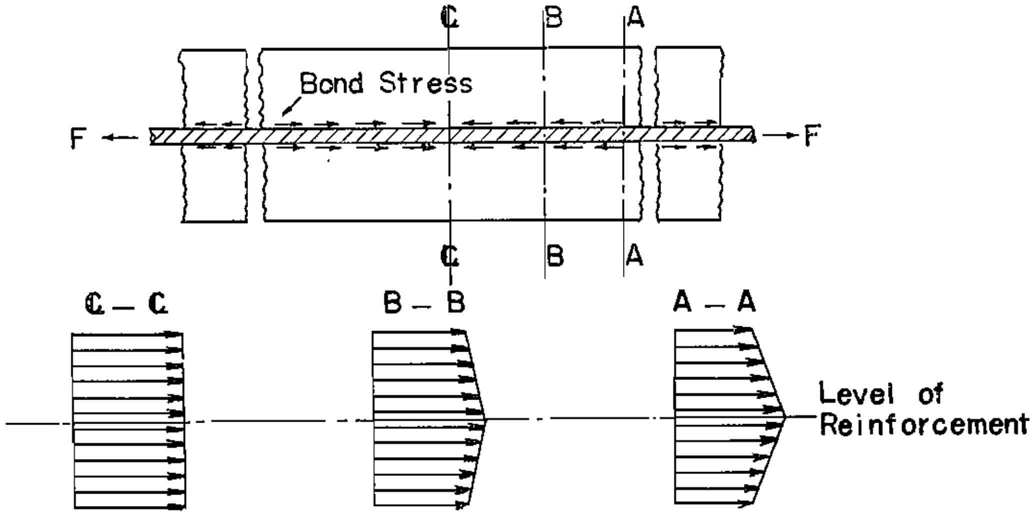
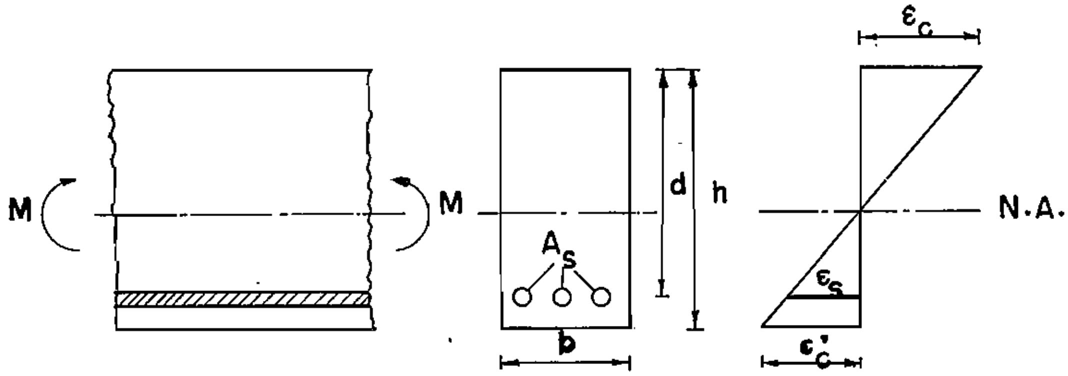
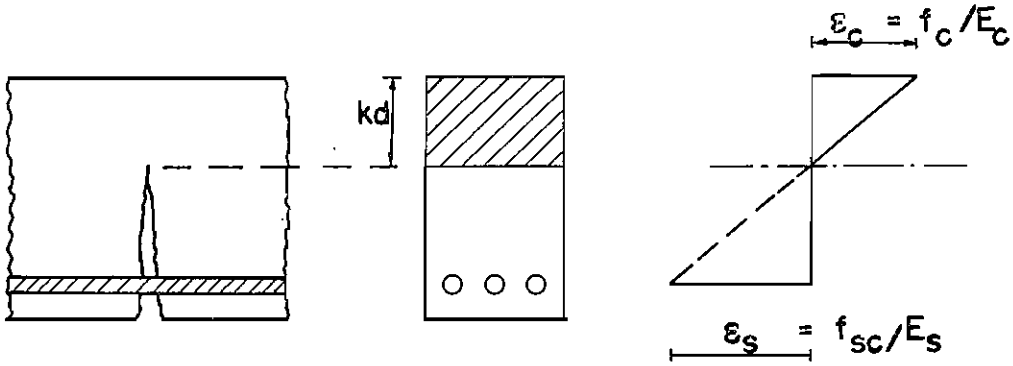


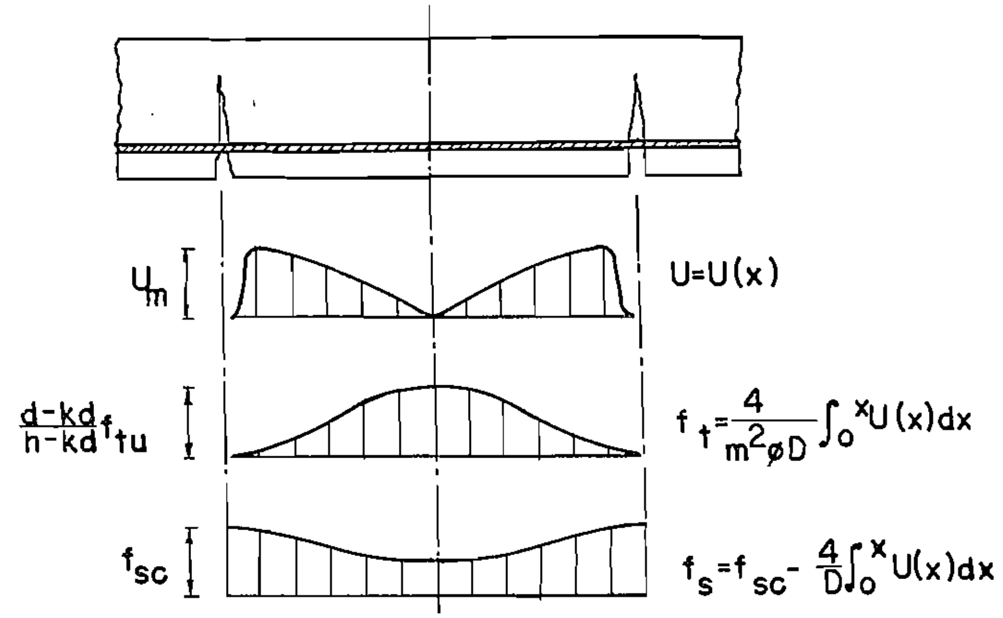
FIG.10, EFFECT OF SHEAR DEFORMATION IN TENSILE STRESS DISTRIBUTION IN CONCRETE IN AXIALLY LOADED MEMBER.



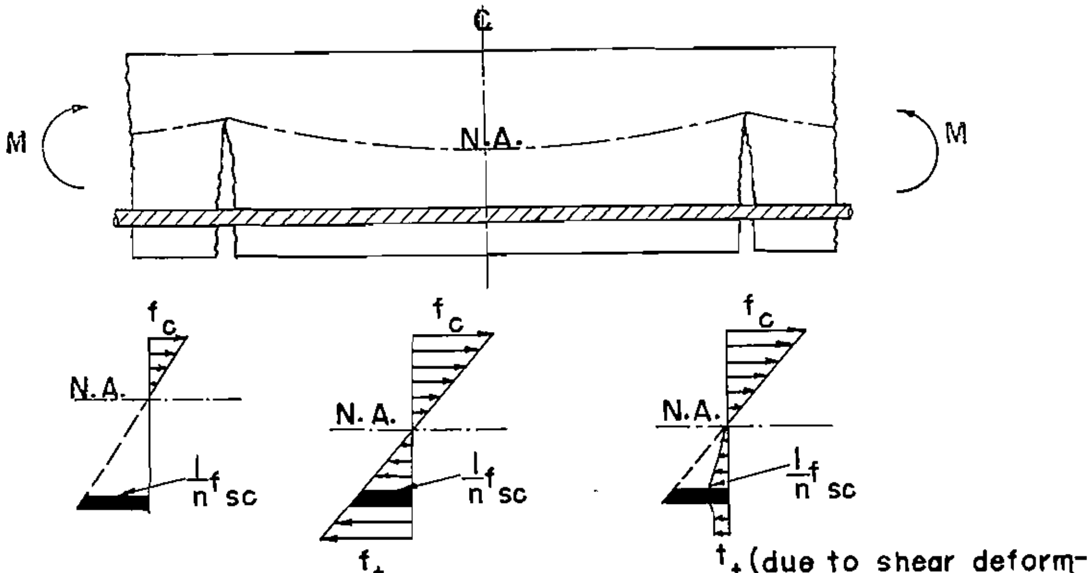
FIG·11, TRANSVERSE STRAIN DISTRIBUTION PRIOR TO THE INITIAL CRCKING IN FLEXURE.



FIG· 12, STRAIN DISTRIBUTION ST CRACKED SECTIONS IN FLEXURE.



FIG·13, LONGITUDINAL STRESS DISTRIBUTION BETWEEN TWO ADJOINING CRAKS. BENDING MOMENT IS CONSTANT.



Cracked Section by Straight-Line-Theory

Center Section by Elastic Theory

Center Section according to Assumption by Chi and kirstein

FIG.14, TRANSVERSE STRESS DISTRIBUTION IN FLEXURE.

港湾技術研究所欧文報告 No.1

1963年2月

編集兼発行人　運輸省港湾技術研究所

発　行　所　運輸省港湾技術研究所
横須賀市川間 162

印　刷　所　中和印刷株式会社
東京都中央区入船町2の3