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The secondary interactions of surface waves.

by

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THE SECONDARY INTERACTIONS OF SURFACE WAVES

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Abstracts: The secondary interactions of surface waves are treated in two roughly divided cases. In Part I, the case in which the first order wave has discrete components is discussed, and in Part II th case in which it makes the continuous spectrum is computed.

In Part I, the main characteristics of the secondary interaction are treated in the one-dimensional problem. They include the case in which the each components of wave have the same direction of progress and the case in which their directions are opposite each other. In the two-dimensional problem, the general expression of the secondary interaction is obtained, and as an example, the clapotis caused by the oblique incidence of progressive wave on the rigid wall is discussed.

A secondary long-crested wave which progresses along the rigid wall is obtained and discussed.

In part II, we treat the wave which does not contain the reverse progressive components. At first, the case of the one-dimensional wave in the finite depth is treated. Numerical results of the second order nonlinearity obtained by this method are in good agreement with the experimental nonlinear effect. Secondly the problem of the two-dimensional progressive wave in the finite depth is computed. A result of the simple application of this computation shows that, if the first order spectrum of wave profile has the sufficient angular spreading, the second order nonlinear components of frequency spectrum will be notably different from those without angular spreading.

Part I. The case of discrete components

I-1. basic equations

The problem is inviscid and irrotational, and the motion is of a type of Stokes. L. J. Tick(1959, 1963)and M.S. Longuet-Higgins & R. W. Stewart(1962)have already shown many aspects of the problem of secondary interaction. In Part I, the author pursues the dynamical behavior of the interaction components and explains the physical meaning of them. Stillmore the problem of clapotic wave is treated with some interesting results.

A brief explanation of basic equations is as follows. We use cartesian co-ordinate, in which x-y plane is horizontal and z axis is taken vertically upwards positive.* u, v and w are the particle velocities in x, y and z direction respectively. The velocity potential φ satisfies the next relation.

$$u=\frac{\partial\varphi}{\partial x},\quad v=\frac{\partial\varphi}{\partial y},\quad w=\frac{\partial\varphi}{\partial z} \qquad (\text{I -1})$$

The equation of continuity is

$$\nabla^2\varphi=0 \qquad \left(\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) \qquad (\text{I -2})$$

The equation of motion can be integrated to

$$\frac{p}{\rho}+gz+\frac{1}{2}q^2+\frac{\partial\varphi}{\partial t}=0 \qquad (\text{I -3})$$

p is the pressure at an arbitrary point in water, and q^2 equals $u^2+v^2+w^2$.

Here we remark that the velocity potential φ contains the second order constant number so as to make the plane $z=0$ consisetnt with mean water level.

Kinematic boundary condition of surface is

$$\frac{\partial\zeta}{\partial t}+u\frac{\partial\zeta}{\partial x}+v\frac{\partial\zeta}{\partial y}=\frac{\partial\varphi}{\partial z} \qquad \text{at} \quad z=\zeta \qquad (\text{I -4})$$

ζ is the surface elevation from mean water level.

Dynamic boundary condition of surface is

$$g\zeta+\frac{1}{2}(q^2)_\zeta+\left(\frac{\partial\varphi}{\partial t}\right)_\zeta=0 \qquad \text{at} \quad z=\zeta \qquad (\text{I -5})$$

Bottom condition is

$$\frac{\partial\varphi}{\partial z}=0 \qquad \text{at } z=-h \qquad (\text{I -6})$$

Here h is the depth to the bottom of water. and is uniform in the considered domain.

The perturbation method is used in concern with wave slope. Using the perturbation of φ, ζ amd p, we put

$$\varphi=\alpha\varphi_1+\alpha^2\varphi_2+\alpha^3\varphi_3+\cdots\cdots \qquad (\text{I -7-1})$$

$$\zeta=\alpha\zeta_1+\alpha^2\zeta_2+\alpha^3\zeta_3+\cdots\cdots \qquad (\text{I -7-2})$$

$$p=p_0+\alpha p_1+\alpha^2 p_2+\alpha^3 p_3+\cdots\cdots \qquad (\text{I -7-3})$$

$$p_0=-\rho gz \qquad (\text{I -7-4})$$

The perturbed equations by α are shown as follows.

From(I — 2)

$$\nabla^2\varphi_1=0,\ \nabla^2\varphi_2=0,\ \nabla^2\varphi_3=0 \qquad (\text{I -8-1, 2, 3})$$

From(I — 3)

$$\frac{p_1}{\rho}+\varphi_{1t}=0 \qquad (\text{I -9-1})$$

$$\frac{p_2}{\rho}+\frac{1}{2}(\varphi_{1x}{}^2+\varphi_{1y}{}^2+\varphi_{1z}{}^2)+\varphi_{2t}=0 \qquad (\text{I -9-2})$$

* In Part II, z axis is taken downwards positive.

$$\frac{p_3}{\rho}+\varphi_{1x}\varphi_{2x}+\varphi_{1y}\varphi_{2y}+\varphi_{1z}\varphi_{2z}+\varphi_{3t}=0 \qquad (\text{I-9-3})$$

From (I-4)

$$\zeta_{1t}=w_1(0) \qquad \text{at} \quad z=0 \quad (\text{I-10-1})$$

$$\zeta_{2t}+u_1(0)\zeta_{1x}+v_1(0)\zeta_{1y}=w_2(0)+w_{1z}(0)\zeta_1 \qquad \text{at} \quad z=0 \quad (\text{I-10-2})$$

$$\zeta_{3t}+u_2(0)\zeta_{1x}+u_{1z}(0)\zeta_1\zeta_{1x}+u_1(0)\zeta_{2x}+v_2(0)\zeta_{1y}+v_{1z}(0)\zeta_1\zeta_{1y}$$
$$+v_1(0)\zeta_{2y}=w_3(0)+w_{1z}(0)\zeta_2+w_{2z}(0)\zeta_1+\frac{w_{1zz}(0)}{2}\zeta_1^2 \qquad \text{at} \quad z=0 \quad (\text{I-10-3})$$

From (I-5)

$$g\zeta_1+\varphi_{1t}(0)=0 \qquad \text{at} \quad z=0 \quad (\text{I-11-1})$$

$$g\zeta_2+\frac{1}{2}u_1^2(0)+\frac{1}{2}v_1^2(0)+\frac{1}{2}w_1^2(0)+\varphi_{2t}(0)+\varphi_{1tz}(0)\zeta_1=0$$
$$\text{at} \quad z=0 \quad (\text{I-11-2})$$

$$g\zeta_3+u_1(0)u_2(0)+u_1(0)u_{1z}(0)\zeta_1+v_1(0)v_2(0)+v_1(0)v_{1z}(0)\zeta_1$$
$$+w_1(0)w_2(0)+w_1(0)w_{1z}(0)\zeta_1+\varphi_{3t}(0)+\varphi_{1tz}(0)\zeta_2+\varphi_{2tz}(0)\zeta_1$$
$$+\frac{1}{2}\varphi_{1tzz}\zeta_1^2=0 \qquad \text{at} \quad z=0 \quad (\text{I-11-3})$$

From (I-6)

$$\varphi_{1z}=0, \quad \varphi_{2z}=0, \quad \varphi_{3z}=0 \qquad \text{at} \quad z=-h \quad (\text{I-12-1,2,3})$$

From (I-10) and (I-11)

$$\varphi_{1tt}(0)+g\varphi_{1z}(0)=0 \qquad \text{at} \quad z=0 \quad (\text{I-13-1})$$

$$\varphi_{2tt}(0)+g\varphi_{2z}(0)=-\frac{1}{2}(u_1^2(0)+v_1^2(0)+w_1^2(0))_t-\varphi_{1ztt}(0)\zeta_1-\varphi_{1zt}(0)\zeta_{1t}$$
$$+gu_1(0)\zeta_{1x}+gv_1(0)\zeta_{1y}-g\varphi_{1zz}(0)\zeta_1 \qquad \text{at} \quad z=0 \quad (\text{I-13-2})$$

$$\varphi_{3tt}(0)+g\varphi_{3z}(0)=gu_2(0)\zeta_{1x}+gu_{1z}(0)\zeta_1\zeta_{1x}+gu_1(0)\zeta_{2x}+gv_2(0)\zeta_{1y}$$
$$+gv_{1z}(0)\zeta_1\zeta_{1y}+gv_1(0)\zeta_{2y}-g\varphi_{1zz}^{(0)}\zeta_2-g\varphi_{2zz}(0)\zeta_1-g\frac{1}{2}\varphi_{1zzz}(0)\zeta_1^2$$
$$-u_{1t}(0)u_2(0)-u_1(0)u_{2t}(0)-u_{1t}(0)u_{1z}(0)\zeta_1-u_1(0)u_{1zt}(0)\zeta_1-u_1(0)u_{1z}(0)\zeta_{1t}$$
$$-v_{1t}(0)v_2(0)-v_1(0)v_{2t}(0)-v_{1t}(0)v_{1z}(0)\zeta_1-v_1(0)v_{1zt}(0)\zeta_1-v_1(0)v_{1z}(0)\zeta_{1t}$$
$$-w_{1t}(0)w_2(0)-w_1(0)w_{2t}(0)-w_{1t}(0)w_{1z}(0)\zeta_1-w_1(0)w_{1zt}(0)\zeta_1-w_1(0)w_{1z}(0)\zeta_{1t}$$
$$-\varphi_{1ztt}(0)\zeta_2-\varphi_{1zt}(0)\zeta_{2t}-\varphi_{2ztt}(0)\zeta_1-\varphi_{2zt}(0)\zeta_{1t}-\frac{1}{2}\varphi_{1zztt}(0)\zeta_1^2-\varphi_{1zzt}(0)\zeta_1\zeta_{1t}$$
$$\text{at} \quad z=0 \quad (\text{I-13-3})$$

Equations (I-13-1, 2, 3) are used for the determination of φ_1, φ_2 and φ_3, and equations (I-11-1, 2, 3) are used to compute ζ_1, ζ_2 and ζ_3. Then the pressure is determined by (I-9-1,2,3).

The simplest case is of the one-dimensional wave in deep water. The three component wave in this case is treated as follows.

The first order wave is

$$\varphi_1=a_1c_1e^{k_1z}\sin\{k_1(x-c_1t)+\varepsilon_1\}+a_2c_2e^{k_2z}\sin\{k_2(x-c_2t)+\varepsilon_2\}$$
$$+a_3c_3e^{k_3z}\sin\{k_3(x-c_3t)+\varepsilon_3\} \qquad (\text{I-14})$$

$$\zeta_1=a_1\cos\{k_1(x-c_1t)+\varepsilon_1\}+a_2\cos\{k_2(x-c_2t)+\varepsilon_2\}$$
$$+a_3\cos\{k_3(x-c_3t)+\varepsilon_3\} \qquad (\text{I-15})$$

$$k_1c_1^2=k_2c_2^2=k_3c_3^2=g \quad (k_1,\ k_2,\ k_3>0) \qquad \text{(I-16)}$$

The second order wave by the interaction is

$$\varphi_2=\frac{2a_1c_1k_1a_2c_2k_2(c_1k_1-c_2k_2)}{(c_1k_1-c_2k_2)^2-g|k_1-k_2|}e^{|k_1-k_2|z}\sin\{(k_1-k_2)x-(c_1k_1-c_2k_2)t+(\varepsilon_1-\varepsilon_2)\}$$
$$+\frac{2a_2c_2k_2a_3c_3k_3(c_2k_2-c_3k_3)}{(c_2k_2-c_3k_3)^2-g|k_2-k_3|}e^{|k_2-k_3|z}\sin\{(k_2-k_3)x-(c_2k_2-c_3k_3)t+(\varepsilon_2-\varepsilon_3)\}$$
$$+\frac{2a_1c_1k_1a_3c_3k_3(c_3k_3-c_1k_1)}{(c_3k_3-c_1k_1)^2-g|k_3-k_1|}e^{|k_3-k_1|z}\sin\{(k_3-k_1)x-(c_3k_3-c_1k_1)t+(\varepsilon_3-\varepsilon_1)\}$$
$$+\text{const. } t \qquad \text{const.}=0 \qquad \text{(I-17)}$$

This may be rewritten as

$$\varphi_2=\begin{matrix}-a_1c_1k_1a_2(k_1>k_2)\\+a_1c_2k_2a_2(k_2>k_1)\end{matrix}e^{|k_1-k_2|z}\sin\{(k_1-k_2)x-(c_1k_1-c_2k_2)t+(\varepsilon_1-\varepsilon_2)\}$$
$$+\begin{matrix}-a_2c_2k_2a_3(k_2>k_3)\\+a_2c_3k_3a_3(k_3>k_2)\end{matrix}e^{|k_2-k_3|z}\sin\{(k_2-k_3)x-(c_2k_2-c_3k_3)t+(\varepsilon_2-\varepsilon_3)\}$$
$$+\begin{matrix}-a_3c_3k_3a_1(k_3>k_1)\\+a_3c_1k_1a_1(k_1>k_3)\end{matrix}e^{|k_3-k_1|z}\sin\{(k_3-k_1)x-(c_3k_3-c_1k_1)t+(\varepsilon_3-\varepsilon_1)\} \qquad \text{(I-17')}$$

ζ_2 is computed as

$$\zeta_2=\frac{a_1^2k_1}{2}\cos\{2k_1(x-c_1t)+2\varepsilon_1\}+\frac{a_2^2k_2}{2}\cos\{2k_2(x-c_2t)+2\varepsilon_2\}$$
$$+\frac{a_3^2k_3}{2}\cos\{2k_3(x-c_3t)+2\varepsilon_3\}+\frac{a_1a_2(k_1+k_2)}{2}\cos\{(k_1+k_2)x-(c_1k_1+c_2k_2)t$$
$$+(\varepsilon_1+\varepsilon_2)\}+\frac{a_2a_3(k_2+k_3)}{2}\cos\{(k_2+k_3)x-(c_2k_2+c_3k_3)t+(\varepsilon_2+\varepsilon_3)\}$$
$$+\frac{a_3a_1(k_3+k_1)}{2}\cos\{(k_3+k_1)x-(c_3k_3+c_1k_1)t+(\varepsilon_3+\varepsilon_1)\}$$
$$-\frac{a_1a_2|k_2-k_1|}{2}\cos\{(k_1-k_2)x-(c_1k_1-c_2k_2)t+(\varepsilon_1-\varepsilon_2)\}$$
$$-\frac{a_2a_3|k_3-k_2|}{2}\cos\{(k_2-k_3)x-(c_2k_2-c_3k_3)t+(\varepsilon_2-\varepsilon_3)\}$$
$$-\frac{a_3a_1|k_1-k_3|}{2}\cos\{(k_3-k_1)x-(c_3k_3-c_1k_1)t+(\varepsilon_3-\varepsilon_1)\} \qquad \text{(I-18)}$$

It is clear by this computation that the principal character of the secondary interaction can be obtained by the computation of the interaction of two components wave. The character of the case of the interaction of three or more components wave is easily deduced from the case of the two components wave.

I-2 the one-dimensional case

Here we compute the case of two components wave of the depth of finite in the one-dimensional case.

The first order wave is

$$\varphi_1=b_1\frac{\cosh k_1(h+z)}{\sinh k_1\ h}\sin k_1(x-c_1t)+b_2\frac{\cosh k_2(h+z)}{\sinh k_2\ h}\sin k_2(x-c_2t) \qquad \text{(I-19)}$$

$$\zeta_1=a_1\cos k_1(x-c_1t)+a_2\cos k_2(x-c_2t) \quad (\text{I-20})$$

$$\left.\begin{array}{l} b_1=a_1c_1,\ b_2=a_2c_2 \\ c_1^2=\dfrac{g}{k_1}\tanh k_1h,\ \ c_2^2=\dfrac{g}{k_2}\tanh k_2h,\ k_1,k_2>0 \end{array}\right\} \quad (\text{I-21})$$

$$p_0=-\rho gz \quad (\text{I-22})$$

$$p_1=\rho b_1k_1c_1\frac{\cosh k_1(h+z)}{\sinh k_1\ h}\cos k_1(x-c_1t)$$

$$+\rho b_2k_2c_2\frac{\cosh k_2(h+z)}{\sinh k_2\ h}\cos k_2(x-c_2t) \quad (\text{I-23})$$

The second order wave is computed as

$$\varphi_2=B_{21}\frac{\cosh 2k_1(h+z)}{\sinh 2k_1\ h}\sin 2k_1(x-c_1t)+B_{22}\frac{\cosh 2k_2(h+z)}{\sinh 2k_2\ h}\sin 2k_2(x-c_2t)$$

$$+B_{23}\frac{\cosh(k_1+k_2)(h+z)}{\sinh(k_1+k_2)h}\sin\{(k_1+k_2)x-(c_1k+c_2k_2)t\}$$

$$+B_{24}\frac{\cosh(k_1-k_2)(h+z)}{\sinh(k_1-k_2)h}\sin\{(k_1-k_2)x-(c_1k_1-c_2k_2)t\}+\text{const. t} \quad (\text{I-24})$$

In (I-24),

$$B_{21}=\frac{-\dfrac{3}{2}b_1^2k_1^3c_1(\coth^2k_1h-1)}{-4k_1^2c_1^2\coth 2k_1h+2k_1g} \quad (\text{I-25-1})$$

$$B_{22}=\frac{-\dfrac{3}{2}b_2^2k_2^3c_2(\coth^2k_2h-1)}{-4k_2^2c_2^2\coth 2k_2h+2k_2g} \quad (\text{I-25-2})$$

$$B_{23}=\frac{b_1b_2k_1k_2(c_1k_1+c_2k_2)(1-\coth k_1h\coth k_2h)-\dfrac{1}{2}b_1k_1^3a_2c_1^2(\coth^2k_1h-1)-\dfrac{1}{2}b_2k_2^3a_1c_2^2(\coth^2k_2h-1)}{-(c_1k_1+c_2k_2)^2\coth(k_1+k_2)h+(k_1+k_2)g} \quad (\text{I-25-3})$$

$$B_{24}=\frac{b_1b_2k_1k_2(c_2k_2-c_1k_1)(1+\coth k_1h\coth k_2h)-\dfrac{1}{2}b_1k_1^3a_2c_1^2(\coth^2k_1h-1)+\dfrac{1}{2}b_2k_2^3a_1c_2^2(\coth^2k_2h-1)}{-(c_1k_1-c_2k_2)^2\coth(k_1-k_2)h+(k_1-k_2)g} \quad (\text{I-25-4})$$

$$\text{const.}=-\frac{1}{4}b_1^2k_1^2(\coth^2k_1h-1)-\frac{1}{4}b_2^2k_2^2(\coth^2k_2h-1) \quad (\text{I-25-5})$$

Putting $\zeta_2=\zeta_{21}+\zeta_{22}+\zeta_{23}+\zeta_{24}$,

$$\zeta_{21}=\frac{1}{g}\left\{2k_1c_1B_{21}\coth 2k_1h+\frac{1}{2}b_1^2k_1^2-\frac{1}{4}b_1^2k_1^2(\coth^2k_1h-1)\right\}\cos 2k_1(x-c_1t) \quad (\text{I-26-1})$$

$$\zeta_{22}=\frac{1}{g}\left\{2k_2c_2B_{22}\coth 2k_2h+\frac{1}{2}b_2^2k_2^2-\frac{1}{4}b_2^2k_2^2(\coth^2k_2h-1)\right\}\cos 2k_2(x-c_2t) \quad (\text{I-26-2})$$

$$\zeta_{23}=\frac{1}{g}\left\{(c_1k_1+c_2k_2)B_{23}\coth(k_1+k_2)h+\frac{1}{2}(k_1^2c_1b_1a_2+k_2^2c_2b_2a_1)\right.$$
$$\left.-\frac{1}{2}b_1b_2k_1k_2(\coth k_1h\coth k_2h-1)\right\}\cos\left\{(k_1+k_2)x-(c_1k_1+c_2k_2)t\right\} \quad (\text{I}-26-3)$$

$$\zeta_{24}=\frac{1}{g}\left\{(c_1k_1-c_2k_2)B_{24}\coth(k_1-k_2)h+\frac{1}{2}(k_1^2c_1b_1a_2+k_2^2c_2b_2a_1)\right.$$
$$\left.-\frac{1}{2}b_1b_2k_1k_2(\coth k_1h\coth k_2h+1)\right\}\cos\left\{(k_1-k_2)x-(c_1k_1-c_2k_2)t\right\} \quad (\text{I}-26-4)$$

Putting $p_2=p_{20}+p_{21}+p_{22}+p_{23}+p_{24}$

$$p_{20}=-\frac{1}{2}\rho b_1^2k_1^2\frac{\sinh^2k_1(h+z)}{\sinh^2 k_1h}-\frac{1}{2}\rho b_2^2k_2^2\frac{\sinh^2k_2(h+z)}{\sinh^2 k_2h} \quad (\text{I}-27-1)$$

$$p_{21}=\left\{2\rho k_1c_1B_{21}\frac{\cosh 2k_1(h+z)}{\sinh 2k_1h}-\frac{1}{4}\rho b_1^2k_1^2\frac{1}{\sinh^2k_1h}\right\}\cos 2k_1(x-c_1t) \quad (\text{I}-27-2)$$

$$p_{22}=\left\{2\rho k_2c_2B_{22}\frac{\cosh 2k_2(h+z)}{\sinh 2k_2h}-\frac{1}{4}\rho b_2^2k_2^2\frac{1}{\sinh^2k_2h}\right\}\cos 2k_2(x-c_2t) \quad (\text{I}-27-3)$$

$$p_{23}=\left\{\rho(c_1k_1+c_2k_2)B_{23}\frac{\cosh(k_1+k_2)(h+z)}{\sinh(k_1+k_2)h}-\frac{1}{2}\rho b_1b_2k_1k_2\frac{\cosh(k_1-k_2)(h+z)}{\sinh k_1h\sinh k_2h}\right\}$$
$$\times\cos\{(k_1+k_2)x-(c_1k_1+c_2k_2)t\} \quad (\text{I}-27-4)$$

Table-1

$c_1,\ c_2>0$

k_2	c_2	a_{21}	a_{22}	a_{23}
0.035	14.79	0.03676	0.04374	0.08028
(0.035)		(0.025)	(0.0175)	(0.0425)
0.040	14.29	0.03676	0.03935	0.07653
(0.040)		(0.025)	(0.020)	(0.0450)
0.045	13.80	0.03676	0.03729	0.07444
(0.045)		(0.025)	(0.0225)	(0.0475)
0.050	13.32	0.03676	0.03676	0.07352
(0.050)		(0.025)	(0.025)	(0.0500)
0.055	12.86	0.03676	0.03665	0.07343
(0.055)		(0.025)	(0.0275)	(0.0525)
0.060	12.44	0.03676	0.03719	0.07404
(0.060)		(0.025)	(0.030)	(0.0550)
0.065	12.03	0.03676	0.03811	0.07509
(0.065)		(0.025)	(0.0325)	(0.0575)

$$p_{24}=\left\{\rho(c_1k_1-c_2k_2)B_{24}\frac{\cosh(k_1-k_2)(h+z)}{\sinh(k_1-k_2)h}-\frac{1}{2}\rho b_1b_2k_1k_2\frac{\cosh(k_1+k_2)(h+z)}{\sinh k_1h\sinh k_2h}\right\}$$
$$\times\cos\{(k_1-k_2)x-(c_1k_1-c_2k_2)t\} \qquad (\text{I}\text{-}27\text{-}5)$$

By this way the computation of the secondary interaction of wave of two components shows that (i) elementary wave has its own second order component as if the single elementary wave exists, and that (ii) stillmore in general the secondary waves which have the wave number equal to the sum and the difference of the original wave numbers exist.

These second order waves are all bound waves,* and not free waves. The bound wave has some different characters from usual free waves. For instance, the wave celerity, the particle velocity and the relation between surface profile and pressure difference are counted up as these different points which do not satisfy the condition of free wave.

Further it should be added to remark that, when the wave components progress to

* In the case of capillary-gravity waves, the second order waves may be possible to contain induced free waves. (L. F. Mc Goldrick (1965))

a_{24}	coefficient of $p_{24(z=0)}$	coefficient of $p_{24(z=-h)}$	$\frac{c_1k_1+c_2k_2}{k_1+k_2}$	$\frac{c_1k_1-c_2k_2}{k_1-k_2}$
−0. 04846	−0. 8782	−0. 4406	13. 92	9. 89
(−0. 0075)				
−0. 04272	−0. 8038	−0. 4048	13. 75	9. 44
(−0. 0050)				
−0. 03832	−0. 7902	−0. 3724	13. 54	9. 00
(−0. 0025)				
			13. 32	8. 65
(0)				
−0. 03322	−0. 7975	−0. 3228	13. 07	8. 26
(−0. 0025)				
−0. 03288	−0. 8226	−0. 3114	12. 84	8. 04
(−0. 005)				
−0. 03188	−0. 8400	−0. 2894	12. 59	7. 73
(−0. 0075)				

reverse direction each other, the property of the secondary interaction becomes quite different from those of same direction of wave progress. We investigate numerically these properties using the next model.

In (I-20) we put $a_1=1m$, $k_1=0.05m^{-1}$, $h=30m$ and $a_2=1m$, $k_2=0.035$, 0.040, 0.045, ……, 0.060, $0.065m^{-1}$, and both cases of c_1, $c_2>0$ (wave components progress in the same direction), and of $c_1>0$, $c_2<0$ (wave components progress in the reverse direction each other) are considered. Coefficients a_{21}, a_{22}, a_{23}, a_{24} [in the expression $\zeta_2=a_{21}\cos 2k_1(x-c_1t)+a_{22}\cos 2k_2(x-c_2t)+a_{23}\cos\{(k_1+k_2)x-(c_1k_1+c_2k_2)t\}+a_{24}\cos\{(k_1-k_2)x-(c_1k_1-c_2k_2)t\}$], and the wave celerities concerned to the a_{23} term and the a_{24} term, and the coefficient of cosine term of p_{24}(given by (I-27-5)) are shown in Table-1 (in the case of c_1, $c_2>0$) and in Table-2 (in the case $c_1>0$, $c_2<0$). Numerical values in round brackets show the case in which the depth of water becomes infinite and other factors are not changed. In these tables the unit of length is taken to metre, so the values of pressure terms take the water head in the metre unit, when they are divided by 9.8.

In Table-1, a_{21}, a_{22} have values usually given by the one component secondary

Table-2

$c_1>0$, $c_2<0$

k_2	c_2	a_{21}	a_{22}	a_{23}
0.035	−14.79	0.03676	0.04374	0.04418
(0.035)		(0.025)	(0.0175)	(0.0425)
0.040	−14.29	0.03676	0.03935	0.04583
(0.040)		(0.025)	(0.020)	(0.045)
0.045	−13.80	0.03676	0.03729	0.04690
(0.045)		(0.025)	(0.0225)	(0.0475)
0.050	−13.32	0.03676	0.03676	0.05024
(0.050)		(0.025)	(0.025)	(0.0500)
0.055	−12.86	0.03676	0.03665	0.05273
(0.055)		(0.025)	(0.0275)	(0.0525)
0.060	−12.44	0.03676	0.03719	0.05539
(0.060)		(0.025)	(0.030)	(0.0550)
0.065	−12.03	0.03676	0.03811	0.05805
(0.065)		(0.025)	(0.0325)	(0.0575)

interaction. a_{23} is positive, and has the value near $a_{21}+a_{22}$. (In deep water, if $a_1=a_2$, $a_{21}+a_{22}=a_{23}$ is shown by (I-18)). This bound wave has the celerity $\frac{c_1k_1+c_2k_2}{k_1+k_2}$ (c_1, $c_2>0$), which is the intermediate value of the celerity of original two component wave. In the expression of frequency spectrum this component situates in the region of high frequency as same as a_{21} and a_{22} components. a_{23} is negative. The wave number is $|k_1-k_2|$, and becomes very small when k_2 approaches k_1. In usual case the celerity becomes near the group velocity of the first order wave (in the limit of $k_2\cong k_1$, it becomes the group velocity). This wave shows low frequency wave which has its trouph in the vicinity of the maximum amplitude of the compound wave of first order, and its crest situates near the minimum amplitude of the compound wave. In this model, at the case of $a_1=a_2=1$m, this second order long wave has the amplitude of about 4cm and it is not so large.

M. S. Longuet-Higgins & R. W. Stewart (1962) pointed out this wave as one of the interpretation of surf beats.* As this wave is a bound wave, it will disappear when a_1 and a_2 vanish. L. J. Tick (1963), in his treatment of the second order frequency spectrum, shows that the second order component rather concentrates in the region of

* There may be another surf beats caused by free long waves.

a_{24}	coefficient of $p_{24(z=0)}$	coefficient of $p_{24(z=-h)}$	$\frac{c_1k_1+c_2k_2}{k_1+k_2}$	$\frac{c_1k_1-c_2k_2}{k_1-k_2}$
−0.003154	−0.3866	−0.6631	1.745	78.91
(−0.0075)				
−0.001392	−0.3987	−0.7456	1.048	123.76
(−0.0050)				
−0.000342	−0.4178	−0.8221	0.473	257.40
(−0.0025)				
0.00	−0.4435	−0.8871	0.000	∞
(0.00)				
−0.000330	−0.4750	−0.9356	−0.393	−274.66
(−0.0025)				
−0.001348	−0.5070	−0.9668	−0.730	−141.24
(−0.005)				
−0.002945	−0.5563	−0.9765	−1.008	−96.53
(−0.0075)				

low frequency. In this model, as shown in Table-1, $|a_{24}|$ takes a figure up one place from the case of deep water. But a_{24}^2 is relatively small if it is compared with $a_{21}^2+a_{22}^2+a_{23}^2$. The similar treatment in the case of continuous spectrum will be refered in Part II. $p_{24}(z=0)$ shows greater amplitude (about 8cm at water head) than the surface amplitude a_{24}. This is a character of this bound wave. At the limit in which k_2 almost equals k_1, it is noticed that a_{24} does not become zero, though in deep water case a_{24} becomes zero. The long wave that propagates with the group velocity of the first order wave exists.

In Table-2, a_{21} and a_{22} are same as those in Table-1. The values of a_{23} become a little smaller than those in Table-1. It is noticeable that its celerity becomes very small, and that at $k_2=0.05$ the celerity becomes zero. So in this case waves become stationary and do not depend on time and the combination of $k_1=k_2$ and $c_1>0$, $c_2<0$ means physically the existence of the complete clapotic wave. The value of $|a_{24}|$ is far smaller than $|a_{24}|$ shown in Table-1, and is generally smaller than those in deep water. Both of $p_{24}(z=0)$ and $p_{24}(z=-h)$ are negative, and in concern with their absolute values the values at the water bottom are greater than the values at the water surface. In short $|p_{24}|$ of the secondary wave in the present case is far greater than the correspond amplitude of surface elevation. Stillmore the celerity of this wave is very large. and becomes infinity (in the mathematical meaning) at $k_2=0.05$. In this case variations concerned with a_{24} and p_{24} become the function of time only, periodic at the twofold frequency of the primary wave. This term actually means the component of the twofold frequency in pressure variation, which is only related to time and has not the corresponding surface disturbance. This phenomenon is explained as an important feature of clapotic wave. When the wave number of the two components has some difference each other, this effect varies gradually as shown in Table-2. This interaction wave may be caught as one of typical bound waves, because of the large variation of wave pressure against the negligible variation of surface profile.

By these computations, if k_1 and k_2 are both positive and are not so different, the interaction wave of wave number difference (k_1-k_2), when the first order waves have the same direction ef progress, becomes a long wave which progresses to the same direction as two first order waves.

In the case of which the to wcomponents of first order wave have reverse direction of the progress each other, it becomes a twofold frequency term of the pressure variation in clapotic wave without any noticeable effect to water surface.

The result of these computations, if it is simplified, gives the strict expression of complete clapotic wave of the second order. It is as follows,

$$\zeta_1=2a_1 \cos k_1x \cos k_1c_1t \qquad (\text{I}-28)$$

$$p_1=2\rho\, b_1k_1c_1\frac{\cosh k_1(h+z)}{\sinh k_1h}\cos k_1x\cos k_1c_1t \qquad (\text{I}-29)$$

$$\zeta_{21}+\zeta_{22}=\left\{\frac{3}{2}a_1{}^2k_1\coth^3k_1h-\frac{1}{2}a_1{}^2k_1\coth k_1h\right\}\cos 2k_1x\cos 2k_1c_1t \qquad (\text{I}-30\text{-}1)$$

$$\zeta_{23}=\frac{1}{2}a_1^2k_1(\coth k_1h+\tanh k_1h)\cos 2k_1x \qquad (\text{I}-30\text{-}2)$$

$$\zeta_{24}=0 \qquad (\text{I}-30\text{-}3)$$

$$p_{20}=-\rho b_1^2k_1^2\frac{\sinh^2k_1(h+z)}{\sinh^2k_1h} \qquad (\text{I}-31\text{-}1)$$

$$p_{21}+p_{22}=\rho b_1^2k_1^2\frac{1}{\sinh^2 k_1h}\left\{\frac{3}{2}\ \frac{\cosh 2k_1(h+z)}{\sinh^2k_1h}-\frac{1}{2}\right\}\cos 2k_1x\cos 2k_1c_1t \qquad (\text{I}-31\text{-}2)$$

$$p_{23}=\frac{1}{2}\rho b_1^2k_1^2\frac{1}{\sinh^2 k_1h}\cos 2k_1x \qquad (\text{I}-31\text{-}3)$$

$$p_{24}=\left\{-\frac{1}{2}\rho b_1^2k_1^2(3+\coth^2k_1h)+\frac{1}{2}\rho b_1^2k_1^2\frac{\cosh 2k_1(h+z)}{\sinh^2k_1h}\right\}\cos 2k_1c_1t \qquad (\text{I}-31\text{-}4)$$

Here relations between b_1, a_1 and c_1 are given by (I-21).

I-3 the two-dimensional case

The two components wave in the case of finite depth is treated two-dimensionally. The first order wave is

$$\varphi_1=b_1\frac{\cosh k_1(h+z)}{\sinh k_1h}\sin(\boldsymbol{K}_1.X-\sigma_1t)+b_2\frac{\cosh k_2(h+z)}{\sinh k_2h}\sin(\boldsymbol{K}_2.X-\sigma_2t) \qquad (\text{I}-32)$$

$$\zeta_1=a_1\cos(\boldsymbol{K}_1.X-\sigma_1t)+a_2\cos(\boldsymbol{K}_2.X-\sigma_2t) \qquad (\text{I}-33)$$

$$p_0=-\rho gz \qquad (\text{I}-34)$$

$$p_1=\rho b_1\sigma_1\frac{\cosh k_1(h+z)}{\sinh k_1h}\cos(\boldsymbol{K}_1.X-\sigma_1t)+\rho b_2\sigma_2\frac{\cosh k_2(h+z)}{\sinh k_2h}\cos(\boldsymbol{K}_2.X-\sigma_2t) \qquad (\text{I}-35)$$

$$\left.\begin{array}{l}\sigma_1^2=gk_1\tanh k_1h,\quad \sigma_2^2=gk_2\tanh k_2h\\ b_1k_1=a_1\sigma_1,\quad b_2k_2=a_2\sigma_2,\quad \sigma_1,\ \sigma_2>0\\ |\boldsymbol{K}_1|=k_1,\quad |\boldsymbol{K}_2|=k_2\end{array}\right\} \qquad (\text{I}-36)$$

The second order wave is computed as

$$\begin{aligned}\varphi_2=&B_{21}\frac{\cosh 2k_1(h+z)}{\sinh 2k_1h}\sin 2(\boldsymbol{K}_1.X-\sigma_1t)+B_{22}\frac{\cosh 2k_2(h+z)}{\sinh 2k_2h}\sin 2(\boldsymbol{K}_2.X-\sigma_2t)\\ &+B_{23}\frac{\cosh|\boldsymbol{K}_1+\boldsymbol{K}_2|(h+z)}{\sinh|\boldsymbol{K}_1+\boldsymbol{K}_2|h}\sin\{(\boldsymbol{K}_1+\boldsymbol{K}_2).X-(\sigma_1+\sigma_2)t\}\\ &+B_{24}\frac{\cosh|\boldsymbol{K}_1-\boldsymbol{K}_2|(h+z)}{\sinh|\boldsymbol{K}_1-\boldsymbol{K}_2|h}\sin\{(\boldsymbol{K}_1-\boldsymbol{K}_2).X-(\sigma_1-\sigma_2)t\}\\ &+\text{const.t}\end{aligned} \qquad (\text{I}-37)$$

$$B_{21}=\frac{1}{2k_1g-4\sigma_1^2\coth 2k_1h}\ \frac{3}{2}\ b_1^2k_1^2\sigma_1(1-\coth^2k_1h) \qquad (\text{I}-38\text{-}1)$$

$$B_{22}=\frac{1}{2k_2g-4\sigma_2^2\coth 2k_2h}\ \frac{3}{2}\ b_2^2k_2^2\sigma_2(1-\coth^2k_2h) \qquad (\text{I}-38\text{-}2)$$

$$\begin{aligned}B_{23}=&\frac{1}{|\boldsymbol{K}_1+\boldsymbol{K}_2|g-(\sigma_1+\sigma_2)^2\coth|\boldsymbol{K}_1+\boldsymbol{K}_2|h}\Big[\frac{1}{2}b_1b_2(\sigma_1+\sigma_2)\{k_1k_2\\ &-\boldsymbol{K}_1.\boldsymbol{K}_2\coth k_1h\coth k_2h\}+\frac{1}{2}a_2b_1k_1^2g(\tanh k_1h-\coth k_1h)\end{aligned}$$

$$+\frac{1}{2}a_1b_2k_2^2g(\tanh k_2h-\coth k_2h)+\frac{1}{2}\sigma_1\sigma_2\,(a_1b_2k_2+a_2b_1k_1)$$

$$-\frac{1}{2}g\boldsymbol{K}_1.\boldsymbol{K}_2(a_1b_2\coth k_2h+a_2b_1\coth k_1h)\Big] \qquad (\text{I}-38\text{-}3)$$

$$B_{24}=\frac{1}{|\boldsymbol{K}_1-\boldsymbol{K}_2|g-(\sigma_1-\sigma_2)^2\coth|\boldsymbol{K}_1-\boldsymbol{K}_2|h}\Big[-\frac{1}{2}b_1b_2(\sigma_1-\sigma_2)\{k_1k_2$$

$$+\boldsymbol{K}_1.\boldsymbol{K}_2\coth k_1h\coth k_2h\}+\frac{1}{2}a_2b_1k_1^2g(\tanh k_1h-\coth k_1h)$$

$$-\frac{1}{2}a_1b_2k_2^2g(\tanh k_2h-\coth k_2h)+\frac{1}{2}\sigma_1\sigma_2(a_1b_2k_2-a_2b_1k_1)$$

$$-\frac{1}{2}g\boldsymbol{K}_1.\boldsymbol{K}_2(a_1b_2\coth k_2h-a_2b_1\coth k_1h)\Big] \qquad (\text{I}-38\text{-}4)$$

$$\text{const.}=\frac{1}{4}b_1^2k_1^2(1-\coth^2k_1h)+\frac{1}{4}b_2^2k_2^2(1-\coth^2k_2h) \qquad (\text{I}-38\text{-}5)$$

Putting the second order wave profile ζ_2 in the form $\zeta_2=\zeta_{21}+\zeta_{22}+\zeta_{23}+\zeta_{24}$,

$$\zeta_{21}=\frac{1}{g}\Big[-\frac{1}{4}b_1^2k_1^2(\coth^2k_1h-1)+2\sigma_1B_{21}\coth 2k_1h+\frac{1}{2}b_1^2k_1^2\Big]$$

$$\times\cos 2(\mathbf{K}_1.X-\sigma_1t) \qquad (\text{I}-39\text{-}1)$$

$$\zeta_{22}=\frac{1}{g}\Big[-\frac{1}{4}b_2^2k_2^2(\coth^2k_2h-1)+2\sigma_2B_{22}\coth 2k_2h+\frac{1}{2}b_2^2k_2^2\Big]$$

$$\times\cos 2(\boldsymbol{K}_2.X-\sigma_2t) \qquad (\text{I}-39\text{-}2)$$

$$\zeta_{23}=\frac{1}{g}\Big[-\frac{1}{2}\{b_1b_2\boldsymbol{K}_1.\boldsymbol{K}_2\coth k_1h\coth k_2h-b_1b_2k_1k_2\}$$

$$+(\sigma_1+\sigma_2)B_{23}\coth|\boldsymbol{K}_1+\boldsymbol{K}_2|h+\frac{1}{2}(\sigma_1b_1k_1a_2+\sigma_2b_2k_2a_1)\Big]$$

$$\times\cos\{(\boldsymbol{K}_1+\boldsymbol{K}_2).X-(\sigma_1+\sigma_2)t\} \qquad (\text{I}-39\text{-}3)$$

$$\zeta_{24}=\frac{1}{g}\Big[-\frac{1}{2}\{b_1b_2\boldsymbol{K}_1.\boldsymbol{K}_2\coth k_1h\coth k_2h+b_1b_2k_1k_2\}$$

$$+(\sigma_1-\sigma_2)B_{24}\coth|\boldsymbol{K}_1-\boldsymbol{K}_2|h+\frac{1}{2}(\sigma_1b_1k_1a_2+\sigma_2b_2k_2a_1)\Big]$$

$$\times\cos\{(\boldsymbol{K}_1-\boldsymbol{K}_2).X-(\sigma_1-\sigma_2)t\} \qquad (\text{I}-39\text{-}4)$$

Putting $p_2=p_{20}+p_{21}+p_{22}+p_{23}+p_{24}$,

$$p_{20}=-\frac{1}{2}\rho b_1^2k_1^2\frac{\sinh^2k_1(h+z)}{\sinh^2k_1h}-\frac{1}{2}\rho b_2^2k_2^2\frac{\sinh^2k_2(h+z)}{\sinh^2k_2h} \qquad (\text{I}-40\text{-}1)$$

$$p_{21}=\Big\{2\rho\sigma_1B_{21}\frac{\cosh 2k_1(h+z)}{\sinh 2k_1h}-\frac{1}{4}\rho b_1^2k_1^2\frac{1}{\sinh^2k_1h}\Big\}\cos 2(\boldsymbol{K}_1.X-\sigma_1t) \qquad (\text{I}-40\text{-}2)$$

$$p_{22}=\Big\{2\rho\sigma_2B_{22}\frac{\cosh 2k_2(h+z)}{\sinh 2k_2h}-\frac{1}{4}\rho b_2^2k_2^2\frac{1}{\sinh^2k_2h}\Big\}\cos 2(\boldsymbol{K}_2.X-\sigma_2t) \qquad (\text{I}-40\text{-}3)$$

$$p_{23}=\Big[\rho(\sigma_1+\sigma_2)B_{23}\frac{\cosh|\boldsymbol{K}_1+\boldsymbol{K}_2|(h+z)}{\sinh|\boldsymbol{K}_1+\boldsymbol{K}_2|h}-\frac{1}{2}\rho b_1b_2\frac{1}{\sinh k_1h\,\sinh k_2h}$$

$$\times\{\boldsymbol{K}_1.\boldsymbol{K}_2\cosh k_1(h+z)\cosh k_2(h+z)-k_1k_2\sinh k_1(h+z)\sinh k_2(h+z)\}\Big]$$

$$\times\cos\{(\boldsymbol{K}_1+\boldsymbol{K}_2).X-(\sigma_1+\sigma_2)t\} \qquad (\text{I}-40\text{-}4)$$

$$p_{24}=\left[\rho(\sigma_1-\sigma_2)B_{24}\frac{\cosh|\boldsymbol{K}_1-\boldsymbol{K}_2|(h+z)}{\sinh|\boldsymbol{K}_1-\boldsymbol{K}_2|h}-\frac{1}{2}\rho b_1b_2\frac{1}{\sinh k_1h\sinh k_2h}\right.$$
$$\left.\times\{\boldsymbol{K}_1.\boldsymbol{K}_2\cosh k_1(h+z)\cosh k_2(h+z)+k_1k_2\sinh k_1(h+z)\sinh k_2(h+z)\}\right]$$
$$\times\cos\{(\boldsymbol{K}_1-\boldsymbol{K}_2).X-(\sigma_1-\sigma_2)t\} \qquad (\text{I}-40-5)$$

A good example of two-dimensional two components wave is a case of the two-dimensional clapotic wave which has an oblique incident angle to a reflective wall. Assuming that the vertical wall situates at $x=0$, we can put

$$\left.\begin{aligned}&a_1=a_2\\&\boldsymbol{K}_1.X=k_1\cos\theta x+k_1\sin\theta y\\&\boldsymbol{K}_2.X=-k_1\cos\theta x+k_1\sin\theta y\\&\boldsymbol{K}_1.\boldsymbol{K}_2=-k_1^2\cos2\theta\\&(\boldsymbol{K}_1+\boldsymbol{K}_2).X=2k_1\sin\theta y\\&(\boldsymbol{K}_1-\boldsymbol{K}_2).X=2k_1\cos\theta x\end{aligned}\right\} \qquad (\text{I}-41)$$

Fig- I -1 shows the relation of incident and reflective waves. Using (I -41), we have

$$\zeta_1=2a_1\cos(k_1\cos\theta x)\cos(k_1\sin\theta y-\sigma_1t) \qquad (\text{I}-42)$$

$$p_0=-\rho gz \qquad (\text{I}-43)$$

$$p_1=2\rho b_1\sigma_1\frac{\cosh k_1(h+z)}{\sinh k_1h}\cos(k_1\cos\theta x)\cos(k_1\sin\theta y-\sigma_1t) \qquad (\text{I}-44)$$

$$\zeta_{21}+\zeta_{22}=\left\{\frac{3}{2}a_1^2k_1\coth^3k_1h-\frac{1}{2}a_1^2k_1\coth k_1h\right\}\cos(2k_1\cos\theta x)$$
$$\times\cos2(k_1\sin\theta y-\sigma_1t) \qquad (\text{I}-45-1)$$

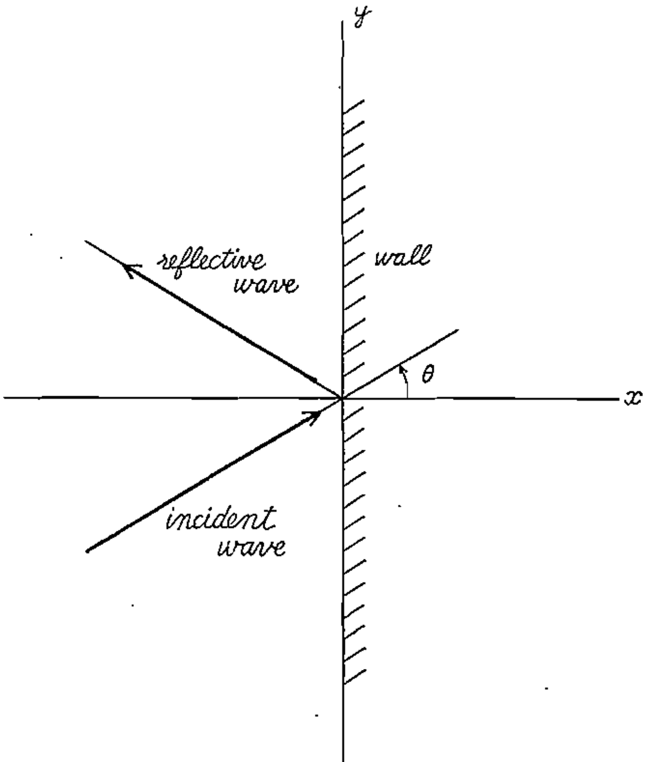

Fig I -1

$$\zeta_{23}=\frac{1}{g}\left[\frac{1}{2}b_1^2k_1^2\{\cos2\theta\coth^2k_1h+3\}+2\sigma_1 B_{23}\coth(2k_1\sin\theta h)\right]$$
$$\times\cos2(k_1\sin\theta y-\sigma_1 t) \qquad \text{(I-45-2)}$$

$$\zeta_{24}=\frac{1}{2}a_1^2k_1\{\cos2\theta\coth k_1h+\tanh k_1h\}\cos2(k_1\cos\theta)x \qquad \text{(I-45-3)}$$

$$p_{20}=-\rho b_1^2k_1^2\frac{\sinh^2k_1(h+z)}{\sinh^2k_1h} \qquad \text{(I-46-1)}$$

$$p_{21}+p_{22}=\frac{\rho b_1^2k_1^2}{\sinh^2 k_1h}\left\{\frac{3}{2}\ \frac{\cosh2k_1(h+z)}{\sinh^2k_1h}-\frac{1}{2}\right\}\cos(2k_1\cos\theta x)$$
$$\times\cos2(k_1\sin\theta y-\sigma_1 t) \qquad \text{(I-46-2)}$$

$$p_{23}=\left[2\rho\sigma_1 B_{23}\frac{\cosh(2k_1\sin\theta)(h+z)}{\sinh(2k_1\sin\theta h)}+\frac{1}{2}\rho\frac{b_1^2k_1^2}{\sinh^2 k_1h}\{\cos2\theta\cosh^2k_1(h+z)\right.$$
$$\left.+\sinh^2k_1(h+z)\}\right]\cos2(k_1\sin\theta y-\sigma_1 t) \qquad \text{(I-46-3)}$$

$$p_{24}=\frac{1}{2}\ \frac{\rho b_1^2k_1^2}{\sinh^2k_1h}\{\cos2\theta\cosh^2k_1(h+z)-\sinh^2k_1(h+z)\}\cos(2k_1\cos\theta x) \qquad \text{(I-46-4)}$$

Here

$$\left.\begin{aligned}B_{23}&=\frac{b_1^2\sigma_1k_1^2}{2k_1\sin\theta g-4\sigma_1^2\coth(2k_1\sin\theta h)}\{3+(2\cos2\theta-1)\coth^2k_1h\}\\ B_{24}&=0\end{aligned}\right\} \qquad \text{(I-47)}$$

It is easily verified that this solution satisfies the boundary condition at $x=0$.

$\zeta_{21}+\zeta_{22}$ and $p_{21}+p_{22}$ show the second order wave peculiar to the incident and reflective first order waves, and they make a group of short-crested waves in accordance with the short-crestedness of $\zeta_1(=\zeta_{11}+\zeta_{12})$. This is usual one and has no special meaning. But we may see the special character of this system of second order wave at a glance of ζ_{23} and p_{23}. ζ_{23} and p_{23} indicate the long-crested wave advancing to (+) y direction along the vertical wall. At the case of $\theta=0$ (incident ray becomes vertical to the wall), ζ_{23} becomes zero, but p_{23} exists and it is a pressure variation of twofold frequency of first order wave. It exactly agrees with p_{24} of (I-31-4). The pressure variation of twofold frequency, which is the function of time only in the case of normal clapotis, thus becomes the long-crested progressive wave along the wall in the case of two-dimensional reflection. An example of behavior of ζ_{23} and p_{23} is shown in Table-3, and the change of ζ_{23} should be marked with the increase of incident angle θ. This second order wave may have some relation to the Mach-reflection problem pointed out by R.L. Wiegel (1964).

ζ_{24} and p_{24} are the steady variation of the local water level (of the second order). They are not dependent on the time, but the relation of ζ_{24} and p_{24} shows that they are the expressions of a exact bound wave. The wave mmbers of ζ_{23} and ζ_{24} are both smaller than $2k_1$ except the limiting case.

Table-3 is a numerical example of the present case. In this table $a_1=1m$, $k_1=0.05\ m^{-1}$ and $h=30m$ are taken, and the incident angle θ is changed from 0 to $\frac{\pi}{2}$. (Terms

Table-3

θ	$k_1 \cos\theta$	$k_1 \sin\theta$	coeff. of ζ_{23}	coeff. of ζ_{24}	coeff. of $p_{23(z=0)}$	coeff. of $p_{24(z=0)}$	coeff. of $p_{23(z=-h)}$	coeff. of $p_{24(z=-h)}$
0°	0. 05	0	0	0. 05025	−0. 4435	0. 04891	−0. 8871	0. 04891
5	0. 04980	0. 00435	−0. 00072	0. 04983	−0. 4506	0. 04480	−0. 8599	0. 04817
10	0. 04924	0. 00868	−0. 00276	0. 04858	−0. 4706	0. 03259	−0. 7857	0. 04596
15	0. 04829	0. 01294	−0. 00536	0. 04655	−0. 4961	0. 01265	−0. 6791	0. 04236
20	0. 04698	0. 01710	−0. 00768	0. 04378	−0. 5187	−0. 01441	−0. 5628	0. 03747
30	0. 04330	0. 02500	−0. 00881	0. 03644	−0. 5299	−0. 08643	−0. 3526	0. 02445
40	0. 03830	0. 03213	−0. 00290	0. 02742	−0. 4720	−0. 1747	−0. 2021	0. 00849
45	0. 03535	0. 03535	0. 00299	0. 02263	−0. 4141	−0. 2217	−0. 1505	0
50	0. 03213	0. 03838	0. 01120	0. 01783	−0. 3329	−0. 2687	−0. 1092	−0. 00849
60	0. 02500	0. 04330	0. 03007	0. 00881	−0. 1488	−0. 3571	−0. 0592	−0. 02445
70	0. 01710	0. 04698	0. 05096	0. 00147	0. 0558	−0. 4291	−0. 0325	−0. 03747
80	0. 00868	0. 04924	0. 06658	−0. 00332	0. 2159	−0. 4761	−0. 0200	−0. 04596
85	0. 00435	0. 04980	0. 07192	−0. 00457	0. 2613	−0. 4883	−0. 0174	−0. 04817
90	0	0. 05	0. 07351	−0. 00499	0. 2768	−0. 4924	−0. 0165	−0. 04891
			coeff. of $\zeta_{21}+\zeta_{22}$		$p_{20(z=0)}=-0.4435$		coeff. of $p_{21(z=-h)}+$	
			0. 07353		coeff. of $p_{21(z=0)}+p_{22(z=0)}$ 0. 2770		$p_{22(z=-h)}$	−0. 01652

of pressure may be expressed by the water head in metre unit, when the correspond values in the table are divided by 9.8). When θ is small, the values in Table-3 are not so different from those of the normal clapotis. But, when θ becomes large, many complicated characters are shown. For instance the signs of ζ_{24} and of $p_{24}(z=0)$ are inverse each other at some values of θ. These properties reveal the intricate characters of the second order bound waves, and they cannot be explained from usual first order free waves.

Part II. The case of continuous spectra

II-1. forwards

The case of continuous spectrum had already been treated one-dimensionally by

L. J. Tick (1959) in the case of deep water.

Concerning the case of the finite depth, he showed the brief explanation in the second paper (1963), but the detailed computation is not published. In this part we make computations of the cases of finite depth in one-dimension and also in two-dimension.

Waves do not contain reverse progressive components as same as in Tick's computation. We use the cartesian co-ordinate x, y, z, in which +x axis is the direction of progress of waves in the one-dimensional treatment, and is the principal direction of progress of waves in the two-dimensional treatment. $z=o$ means the mean water level in the first approximation, and z axis is positive downward. (this is inverse to z axis in Part I, and is taken as same as in Tick's computation.) $z=d$ means the water bottom.

The author found, in the model experiment of development of wind waves, that the nonlinear component of the frequency spectrum appears at the region of about two-fold frequency of the spectrum peak in the case of finite depth of water. This is same in the case of deep water. As our method of spectrum analysis in the model experiment is analogue type, there is some doubt that this nonlinear component is strengthened by the nonlinearity of the analyser itself. To know the accurate phenomena of the nonlinear effect caused by the dynamical and kinematical surface boundary conditions, we need this computation of second order nonlinearity of continuous spectrum. Throughout this computation we also assume the independence of the first order component waves each other. This is one of the important assumptions in this problem and seems to be consistent with the actual states of the sea.

II-2. one-dimensional case

The first order progressive wave is presented by Fourier-Stieltjes type expression as one of real stationary process.

$$\phi^{(1)}=\int_{-\infty}^{\infty} e^{i[-sgn(\omega)|F(\omega)|x+\omega t]}d\zeta_1(\omega)\cosh F(\omega)(z-d) \qquad (\text{II-1})$$

$$\eta^{(1)}=i\int_{-\infty}^{\infty}\frac{1}{\omega}e^{i[-sgn(\omega)|F(\omega)|x+\omega t]}d\zeta_1(\omega)\sinh F(\omega)d\cdot F(\omega) \qquad (\text{II-2})$$

Here $\omega^2=gF(\omega)\tanh F(\omega)d$ (II-3)

For convenience of computation, we consider the case in which $F(\omega)$ has the same sign with ω.

Putting $\frac{i}{\omega}d\zeta_1(\omega)\sinh F(\omega)d\cdot F(\omega)=d\zeta_2(\omega)$ (II-4)

, (II-1) and (II-2) are presented by

$$\phi^{(1)}=\int_{-\infty}^{\infty} e^{i[-sgn(\omega)|F(\omega)|x+\omega t]}d\zeta_2\frac{(-)i\omega\cosh F(\omega)(z-d)}{\sinh F(\omega)d\cdot F(\omega)} \tag{II-5}$$

$$\eta^{(1)}=\int_{-\infty}^{\infty} e^{i[-sgn(\omega)|F(\omega)|x+\omega t]}d\zeta_2(\omega) \tag{II-6}$$

$\phi^{(2)}$ should be determined by

$$\phi_{tt}{}^{(2)}-g\phi_z{}^{(2)}=\phi_{zz}{}^{(1)}\phi_t{}^{(1)}-2\phi_x{}^{(1)}\phi_{tx}{}^{(1)}-\phi_z{}^{(1)}\phi_{zt}{}^{(1)}-\frac{1}{g}(\phi_{tz}{}^{(1)}\phi_{tt}{}^{(1)}+\phi_{ttz}{}^{(1)}\phi_t{}^{(1)})$$

$$\text{at}\quad z=0 \tag{II-7}$$

Using (II-5), the right hand side of this perturbed equation is computed. We obtain

$$[\phi_{zz}{}^{(1)}\phi_t{}^{(1)}]_{z=0}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i[-\{sgn(\omega)|F(\omega)|+sgn(\omega')|F(\omega')|\}x+(\omega+\omega')t]}\frac{d\zeta_2(\omega)d\zeta_2(\omega')}{2}$$

$$\times\Big[\frac{(-1)i\omega\omega'^2F(\omega)\cosh F(\omega)d\cdot\cosh F(\omega')d}{\sinh F(\omega)d\cdot\sinh F(\omega')d\cdot F(\omega')}$$

$$+\frac{(-1)i\omega'\omega^2F(\omega')\cosh F(\omega')d\cdot\cosh F(\omega)d}{\sinh F(\omega')d\cdot\sinh F(\omega)d\cdot F(\omega)}\Big] \tag{II-8}$$

But this expression is rather tedious, and so we use the expression

$$[\phi_{zz}{}^{(1)}\phi_t{}^{(1)}]_{z=0}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i[-\{sgn(\omega)|F(\omega)|+sgn(\omega')|F(\omega')|\}x+(\omega+\omega')t]}d\zeta_2(\omega)d\zeta_2(\omega')$$

$$\times\frac{(-1)i\omega\omega'^2F(\omega)\cosh F(\omega)d\cdot\cosh F(\omega')d}{\sinh F(\omega)d\cdot\sinh F(\omega')d\cdot F(\omega')} \tag{II-9}$$

, and other terms are treated in the same way. We use the symmetrical form of the expression at the final form.

Computing the right hand side of (II-7), we obtain

$$(\phi_{tt}{}^{(2)}-g\,\phi_z{}^{(2)})_{z=0}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i\omega\omega'^2d\zeta_2(\omega)d\zeta_2(\omega')e^{i[-\{F(\omega)+F(\omega')\}x+(\omega+\omega')t]}$$

$$\times\Big[\frac{-F(\omega)\coth F(\omega)d\cdot\coth F(\omega')d}{F(\omega')}-2\coth F(\omega)d\cdot\coth F(\omega')d+1+\frac{\omega(\omega'+\omega)}{\omega'^2}\Big] \tag{II-10}$$

Here

$$\left.\begin{aligned}&\text{sgn}(\omega)|F(\omega)|+\text{sgn}(\omega')|F(\omega')|=F(\omega)+F(\omega')\\&\text{sgn}(\omega)\,\text{sgn}(\omega')\frac{|F(\omega)||F(\omega')|}{F(\omega)F(\omega')}=1\end{aligned}\right\} \tag{II-11}$$

are used under the arrangement of the same sign between F(ω) and ω. Using the bottom boundary condition, we put

$$\phi^{(2)}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i[-\{F(\omega)+F(\omega')\}x+(\omega+\omega')t]}d\zeta_2(\omega)d\zeta_2(\omega')G(\omega,\omega')$$

$$\times\cosh(F(\omega)+F(\omega'))(z-d)+\text{const. t} \tag{II-12}$$

Inserting $\phi^{(2)}$ into (II-10), G(ω, ω') is determined. From here we put F(ω)$=k$

for simplicity.

$$G(\omega,\omega')=\frac{i\omega\omega'^2\left[-\frac{k\coth kd\coth k'd}{k'}-2\coth kd\coth k'd+1+\frac{\omega(\omega'+\omega)}{\omega'^2}\right]}{-(\omega+\omega')^2\cosh(k+k')d+g(k+k')\sinh(k+k')d} \quad (\text{II-13})$$

Then we determine $\eta^{(2)}$ by

$$\eta^{(2)}=\frac{1}{g}\phi_t{}^{(2)}+\frac{1}{2g}\phi_z{}^{(1)}\phi_z{}^{(1)}+\frac{1}{2g}\phi_x{}^{(1)}\phi_x{}^{(1)}+\frac{1}{g^2}\phi_{tz}{}^{(1)}\phi_t{}^{(1)} \quad (\text{II-14})$$

It is clear that const. t of $\phi^{(2)}$ relates to the constant term of $\eta^{(2)}$, and so it situates to the zero frequency of frequency spectrum with no influence on the concrete configuration of the spectrum. By this reason we abbreviate the existence of const. t of $\phi^{(2)}$ in the following computations.

Computing the right hand side of (II-14),

$$\eta^{(2)}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i[-(k+k')x+(\omega+\omega')t]}\left[\frac{gkk'}{2\omega\omega'}-\frac{\omega\omega'}{2g}-\frac{\omega^2}{g}\right.$$
$$\left.+\frac{(\omega+\omega')\left[\frac{gk^2}{\omega}+\frac{2kk'g}{\omega}-\frac{\omega'^2\omega}{g}-(\omega'+\omega)\frac{\omega^2}{g}\right]}{g(k+k')\tanh(k+k')d-(\omega+\omega')^2}\right]d\zeta_2(\omega)\,d\zeta_2(\omega') \quad (\text{II-15})$$

We rewrite (II-15) as

$$\eta^{(2)}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i[-(k+k')x+(\omega+\omega')t]}\left[\frac{gkk'}{2\omega\omega'}+\frac{\omega\omega'}{2g}-\frac{(\omega+\omega')^2}{2g}\right.$$
$$\left.+\frac{(\omega+\omega')^2\left\{\frac{1}{2}\,\frac{g\omega'k^2+g\omega k'^2}{\omega\omega'(\omega+\omega')}+\frac{gkk'}{\omega\omega'}+\frac{\omega\omega'}{2g}-\frac{(\omega+\omega')^2}{2g}\right\}}{g|k+k'|\tanh|k+k'|d-(\omega+\omega')^2}\right]d\zeta_2(\omega)\,d\zeta_2(\omega')$$

(II-16)

(II-16) corresponds to relations (9) and (10) in Tick's computation (1963). Using the assumption that the first order wave is independent each other, the frequency spectrum of the second order wave is expressed by

$$\left.\begin{aligned}&S^{(2)}(\lambda)=\int_{-\infty}^{\infty}K(\omega,\lambda)S^{(1)}(\lambda-\omega)S^{(1)}(\omega)\,d\omega\\&\omega+\omega'=\lambda\end{aligned}\right\} \quad (\text{II-17})$$

Here

$$K(\omega,\lambda)=\frac{1}{4}\left[\frac{gk\,\underline{k(\lambda-\omega)}}{\omega(\lambda-\omega)}+\frac{\omega(\lambda-\omega)}{g}-\frac{\lambda^2}{g}\right.$$
$$\left.+\frac{\lambda^2\left\{\frac{g(\lambda-\omega)k^2+g\omega\,\underline{k^2(\lambda-\omega)}}{\omega(\lambda-\omega)\lambda}+\frac{2gk\,\underline{k(\lambda-\omega)}}{\omega(\lambda-\omega)}+\frac{\omega(\lambda-\omega)}{g}-\frac{\lambda^2}{g}\right\}}{g|k+\underline{k(\lambda-\omega)}|\tanh|k+\underline{k(\lambda-\omega)}|d-\lambda^2}\right]^2 \quad (\text{II-18})^*$$

* In Tick's computation (1959, 1963) coefficient $\frac{1}{4}$ is taken to $\frac{1}{2}$ in (II-18), but $\frac{1}{4}$ may be correct. $\underline{k(\lambda-\omega)}$ means k as a function of $\lambda-\omega$

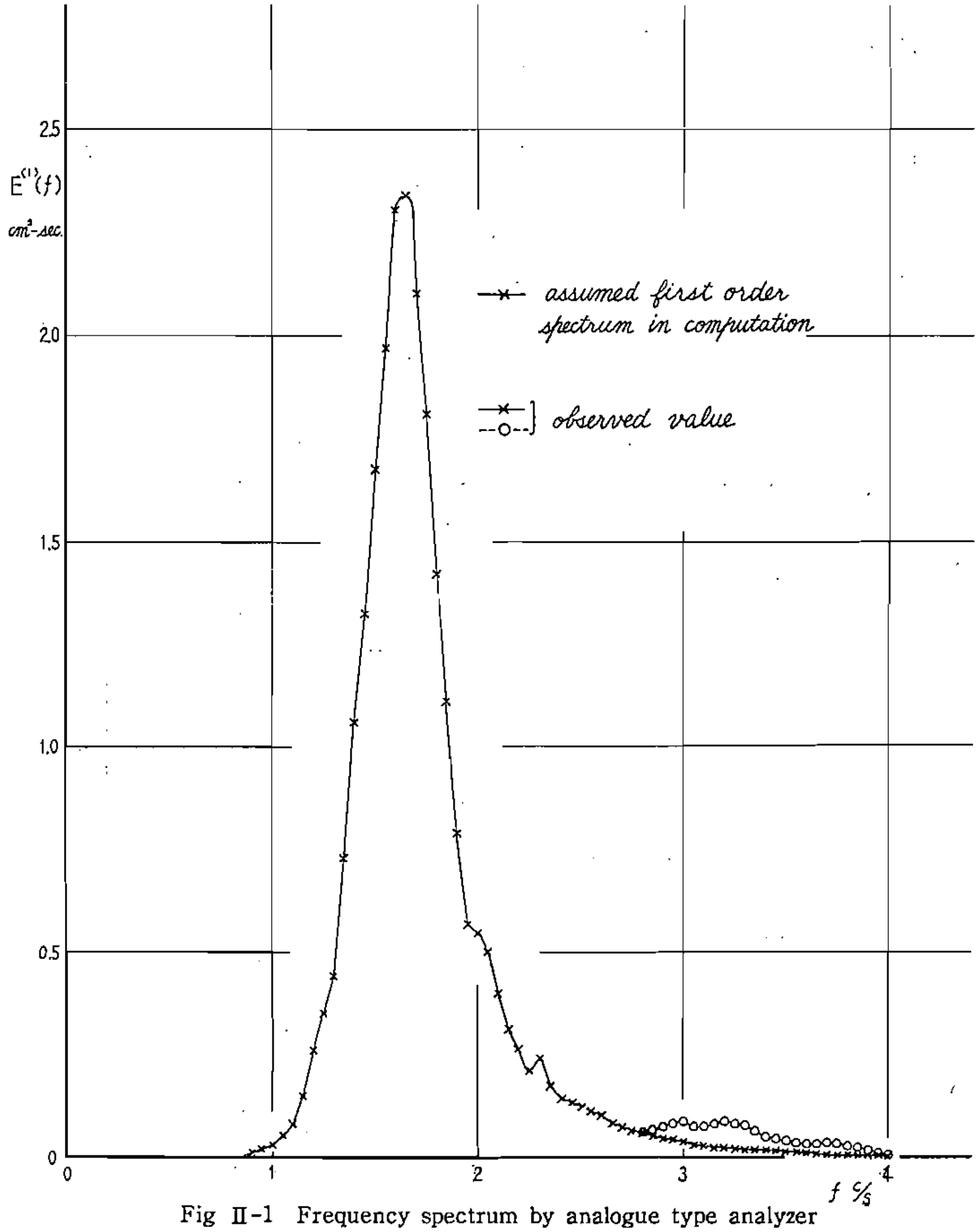

Fig II-1 Frequency spectrum by analogue type analyzer

If we wish to use $f\left(=\frac{\omega}{2\pi}\right)$ instead of λ,

$$<\eta^{(2)2}>=\int S^{(2)}(\lambda)d\lambda=\int E^{(2)}(f_\lambda)df_\lambda$$

$$<\eta^{(1)2}>=\int S^{(1)}(\omega)d\omega=\int E^{(1)}(f)df$$

so

$$E^{(2)}(f_\lambda)=2\pi S^{(2)}(\lambda)=4\pi^2\int_{-\infty}^{\infty}K(\omega,\lambda)S^{(1)}(\lambda-\omega)S^{(1)}(\omega)df$$

$$=4\pi^2\int_{-\infty}^{\infty}K(\omega,\lambda)\frac{E^{(1)}(f_\lambda-f)}{2\pi}\frac{E^{(1)}(f)}{2\pi}df$$

$$=\int_{-\infty}^{\infty}K(\omega,\lambda)E^{(1)}(f_\lambda-f)E^{(1)}(f)df \qquad (\text{II-19})$$

We can use (II-19) instead of (II-17).

Fig-II-1 is an example of the frequency spectrum of wind waves in an experimental tank, and in this case the depth of water to the bottom is 15cm. So the waves must be treated as one of the case of the finite water depth. The spectrum intensity corresponds the(+)side distribution when the spectrum distributes from $-\infty$ to $+\infty$. The x^2—freedom of this frequency spectrum is very large and it is sufficiently reliable. As this spectrum is obtained electrically by the frequency analysis of analogue type (W. J. Pierson Jr. (1954)), the nonlinear components obtained may include some mechanical influence,

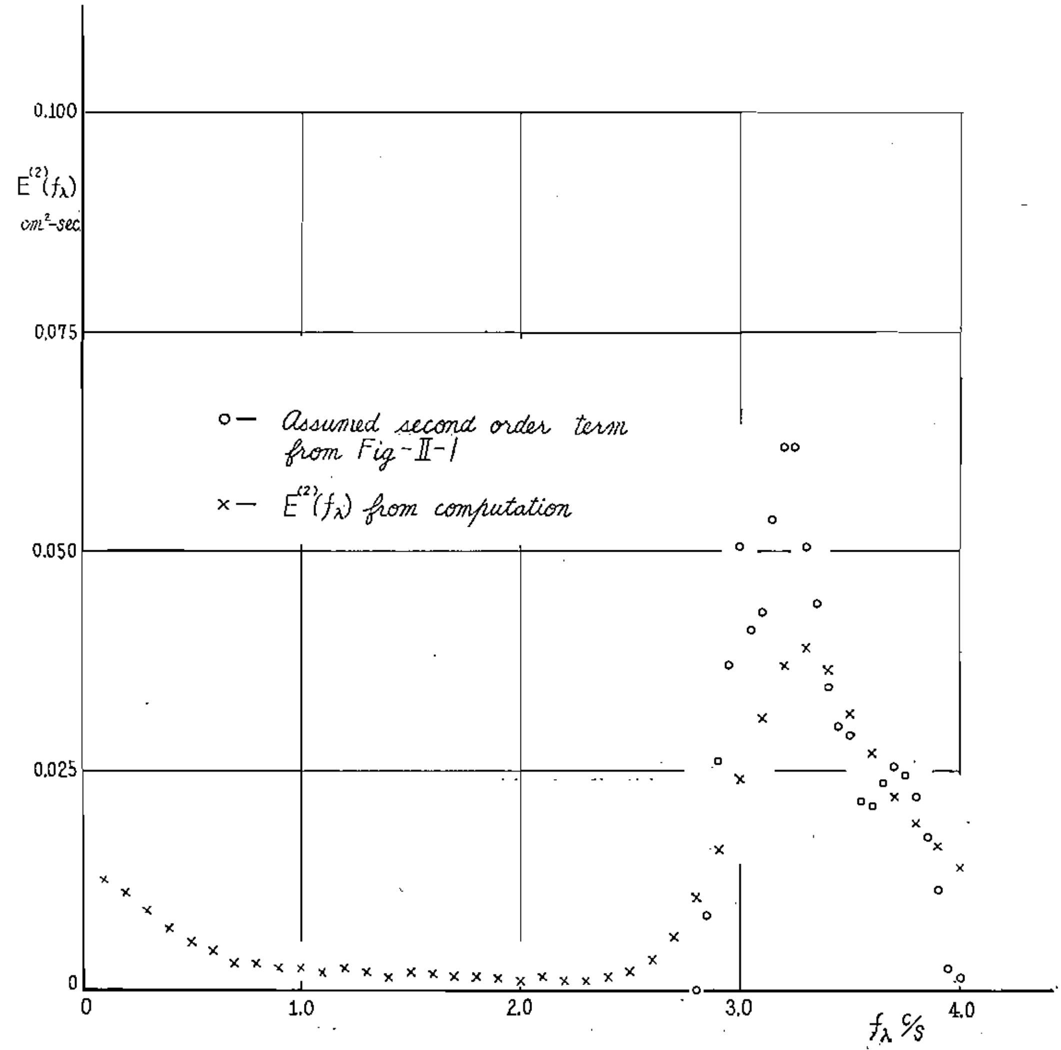

Fig II-2 The second order spectrum $E^{(2)}(f_\lambda)$

and we intend to check the degree of this influence using the computation of this paragraph.

As shown in Fig-II-1 we make the model of $E^{(1)}(f)$ of this spectrum excluding the part which appears to be the nonlinear part clearly. By the numerical computation of (II-18) and (II-19), we find $E^{(2)}(f_l)$ in this case. The result is shown in Fig-II-2. The low frequency second order part which is not found at the analogue determination is clearly recognized. The other nonlinear part appears at the about twofold frequency of the peak frequency, and is very similar to that obtained by analogue analysis. In some places of this part the intensity determined by analogue type is greater, and some mechanical influence may be effective. But its degree is not large. (The second order spectrum of wind waves in experimental waterway may include the effect of angular spreading of waves, and this effect may be infered later.)

When Fig-II-2 is compared with Fig-4-1-2 of L. J. Tick (1963), our result may have some different property from those obtained by Tick.

II-3 the spectrum of wave pressure

The secondary interaction component may be also found in the spectrum of the pressure record of pressure type wave height metre, and it is obvious that the relation of the surface secondary component of wave profile spectrum with the second order component of the pressure spectrum in water is not same with the simple formula (W. J. Pierson Jr. (1954) of the first order component. The new relation for the second order component is examined here.

From the integration of dynamic equation

$$\frac{p^{(0)}}{\rho}=gz \qquad \text{(II-20)}$$

$$\frac{p^{(1)}}{\rho}=-\phi_t^{(1)} \qquad \text{(II-21)}$$

$$\frac{p^{(2)}}{\rho}=-\phi_t^{(2)}-\frac{1}{2}\phi_x^{(1)2}-\frac{1}{2}\phi_z^{(1)2} \qquad \text{(II-22)}$$

From these relations, the spectrum of $p^{(2)}$ is determined. Because we neglect const. t of $\phi^{(2)}$, the spectrum of $p^{(2)}$ has an uncertainty at the component of zero frequency.

As the result of computations,

$$p^{(1)}=-\rho\int_{-\infty}^{\infty} e^{i[-\mathrm{sgn}(\omega)|k|x+\omega t]}d\zeta_2(\omega)\frac{g\cosh k(z-d)}{\cosh kd} \qquad \text{(II-23)}$$

$$=\int_{-\infty}^{\infty} e^{i[-\mathrm{sgn}(\omega)|k|x+\omega t]}d\zeta_4(\omega) \qquad \text{(II-24)}$$

$$p^{(2)}=\rho\Big[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i[-\{\mathrm{sgn}(\omega)|k|+\mathrm{sgn}(\omega')|k'|\}x+(\omega+\omega')t]}d\zeta_2(\omega)\,d\zeta_2(\omega')$$

$$\times\left\{\frac{(\omega+\omega')\left[-\frac{g^2k^2}{\omega}-2\frac{g^2kk'}{\omega}+\omega\omega'^2+\omega^2(\omega'+\omega)\right]\cosh(k+k')(z-d)}{-(\omega+\omega')^2\cosh(k+k')d+g(k+k')\sinh(k+k')d}\right.$$

$$\left.-\frac{1}{2}\omega\omega'\frac{\cosh(k-k')(z-d)}{\sinh kd\sinh k'd}\right\}\Bigg] \qquad (\text{II-25})$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i[-(k+k')x+(\omega+\omega')t]}\frac{d\zeta_4(\omega)\,d\zeta_4(\omega')\cosh kd\cosh k'd}{\rho g^2\cosh k(z-d)\cosh k'(z-d)}$$

$$\times\left[\frac{(\omega+\omega')^2\left[-\frac{g^2}{2}\frac{k^2\omega'+k'^2\omega}{\omega\omega'(\omega+\omega')}-g^2\frac{kk'}{\omega\omega'}-\frac{\omega\omega'}{2}+\frac{(\omega+\omega')^2}{2}\right]\cosh(k+k')(z-d)}{-(\omega+\omega')^2\cosh(k+k')d+g(k+k')\sinh(k+k')d}\right.$$

$$\left.-\frac{1}{2}\omega\omega'\frac{\cosh(k-k')(z-d)}{\sinh kd\sinh k'd}\right] \qquad (\text{II-26})$$

Accordingly, the second order pressure spectrum is given by

$$\left.\begin{aligned}&S_p^{(2)}(\lambda)=\int_{-\infty}^{\infty}K_p(\omega,\lambda)S_p^{(1)}(\lambda-\omega)S_p^{(1)}(\omega)\,d\omega\\&\omega+\omega'=\lambda\end{aligned}\right\} \qquad (\text{II-27})$$

Here

$$K_p(\omega,\lambda)=\frac{1}{4}\left[\frac{\cosh kd\ \cosh \underline{k(\lambda-\omega)}d}{\rho g^2\cosh k(z-d)\cosh\underline{k(\lambda-\omega)}(z-d)}\left\{\frac{\lambda^2\left[-g^2\frac{k^2(\lambda-\omega)+\underline{k^2(\lambda-\omega)}\,\omega}{\omega(\lambda-\omega)\lambda}\right.}{-\lambda^2\cosh(k+\underline{k(\lambda-\omega)})d}\right.\right.$$

$$\frac{\left.-g^2\frac{2k\,\underline{k(\lambda-\omega)}}{\omega(\lambda-\omega)}-\omega(\lambda-\omega)+\lambda^2\right]}{+g(k+\underline{k(\lambda-\omega)})\sinh(k+\underline{k(\lambda-\omega)})d}\cosh(k+\underline{k(\lambda-\omega)})(z-d)$$

$$\left.\left.-\omega(\lambda-\omega)\frac{\cosh(k-k(\lambda-\omega))(z-d)}{\sinh kd\sinh\underline{k(\lambda-\omega)}d}\right\}\right]^2 \qquad (\text{II-28})$$

When we know the first order spectrum of wave pressure at arbitrary point in water*, we can compute its second order spectrum by making use of (II-27)and(II-28). If we use the unit of water head, (II-27) is transformed to

$$\frac{S_p^{(2)}(\lambda)}{\rho^2g^2}=\int_{-\infty}^{\infty}\rho^2g^2K_p(\omega,\lambda)\frac{S_p^{(1)}(\lambda-\omega)}{\rho^2g^2}\ \frac{S_p^{(1)}(\omega)}{\rho^2g^2}d\omega \qquad (\text{II-29})$$

On the other hand, the spectrum of wave profile at the surface is

$$S^{(2)}(\lambda)=\int_{-\infty}^{\infty}K(\omega,\lambda)S^{(1)}(\lambda-\omega)S^{(1)}(\omega)\,d\omega$$

$$=\int_{-\infty}^{\infty}K(\omega,\lambda)\frac{S_p^{(1)}(\lambda-\omega)}{\rho^2g^2}\ \frac{\cosh^2\underline{k(\lambda-\omega)}d}{\cosh^2\underline{k(\lambda-\omega)}(z-d)}\ \frac{S_p^{(1)}(\omega)}{\rho^2g^2}\ \frac{\cosh^2kd}{\cosh^2k(z-d)}\,d\omega \qquad (\text{II-30})$$

Here we applied the first order relation of conversion as

* Actual spectrum always contains the higher order interaction, so the determination in this case is a trial.

$$\frac{S^{(1)}(\omega)}{\frac{S_p^{(1)}(\omega)}{\rho^2 g^2}} = \frac{\cosh^2 kd}{\cosh^2 k(z-d)} \qquad (\text{II-31})$$

Combining (II-29) and (II-30), we can obtain the second order relation of conversion between wave profile and wave pressure as follows.

$$\frac{S^{(2)}(\lambda)}{\frac{S_p^{(2)}(\lambda)}{\rho^2 g^2}} = \frac{\int_{-\infty}^{\infty} K(\omega,\lambda)\, S_p^{(1)}(\lambda-\omega)\frac{\cosh^2 k(\lambda-\omega)d}{\cosh^2 k(\lambda-\omega)(z-d)} S_p^{(1)}(\omega)\frac{\cosh^2 kd}{\cosh^2 k(z-d)}d\omega}{\int_{-\infty}^{\infty} \rho^2 g^2 K_p(\omega,\lambda) S_p^{(1)}(\lambda-\omega) S_p^{(1)}(\omega)d\omega} \qquad (\text{II-32})$$

$K(\omega, \lambda)$, $K_p(\omega, \lambda)$ are already given by (II-18) and (II-28). (II-32) is a very complicated relation in comparison with the first order relation (II-31). This may be one reason of the fact that we cannot treat successfully the high frequency part of the wave profile spectrum obtained in conversion from the bottom pressure spectrum by the relation of (II-31) only. The existence of noise in the pressure spectrum may add the difficulty to this problem.

II-4 two-dimensional case

The two-dimensional case is treated as follows.

$$u=\varphi_x, \qquad v=\varphi_y, \qquad w=\varphi_z \qquad (\text{II-33})$$

$$\phi_{tt}^{(1)} - g\phi_z^{(1)} = 0 \qquad \text{at } z=0 \qquad (\text{II-34})$$

$$\eta_t^{(1)} = \phi_z^{(1)} \qquad \text{at } z=0 \qquad (\text{II-35})$$

$$\phi_{tt}^{(2)} - g\phi_z^{(2)} = \phi_{zz}^{(1)}\phi_t^{(1)} - 2\phi_x^{(1)}\phi_{tx}^{(1)} - 2\phi_y^{(1)}\phi_{ty}^{(1)} - \phi_z^{(1)}\phi_{tz}^{(1)} - \frac{1}{g}(\phi_{tz}^{(1)}\phi_{tt}^{(1)} + \phi_{ttz}^{(1)}\phi_t^{(1)}) \qquad \text{at } z=0 \qquad (\text{II-36})$$

$$\eta^{(2)} = \frac{1}{g}\phi_t^{(2)} + \frac{1}{2g}\phi_z^{(1)}\phi_z^{(1)} + \frac{1}{2g}\phi_x^{(1)}\phi_x^{(1)} + \frac{1}{2g}\phi_y^{(1)}\phi_y^{(1)} + \frac{1}{g^2}\phi_{tz}^{(1)}\phi_t^{(1)} \qquad \text{at } z=0 \qquad (\text{II-37})$$

$$\phi_z^{(1)} = 0 \qquad \text{at } z=d \qquad (\text{II-38})$$

$$\phi_z^{(2)} = 0 \qquad \text{at } z=d \qquad (\text{II-39})$$

Using (II-38), we put

$$\phi^{(1)}(x, y, z, t) = \int_{-\infty}^{\infty} e^{i(\alpha_1 x + \alpha_2 y + \beta t)} d\zeta(\alpha_1, \alpha_2, \beta)\cosh\gamma(z-d) \qquad (\text{II-40})$$

By the relation $\nabla^2\phi^{(1)}=0$, $-\alpha_1^2-\alpha_2^2+\gamma^2=0$ is introduced, and, if we put $\gamma>0$ in this relation, $|K|=k$, $\alpha_1=k\cos\alpha$, $\alpha_2=k\sin\alpha$ and $\gamma=k$ are formed. So (II-40) may be rewritten as

$$\phi^{(1)}(x, y, z, t) = \int_{-\infty}^{\infty} e^{i(K.X+\beta t)} d\zeta(K,\beta)\cosh|K|(z-d) \qquad (\text{II-41})$$

From the condition of real stationary process,

$$e^{i\beta t}d\zeta(\boldsymbol{K},\beta)\cosh|\boldsymbol{K}|(z-d)=[e^{i\beta(-\boldsymbol{K})t}d\zeta(-\boldsymbol{K},\beta(-\boldsymbol{K}))\cosh|\boldsymbol{K}|(z-d)]^* \quad (\text{II-42})$$

Asterisk means the complex conjugate. From (II-34)

$$\beta^2=g|\boldsymbol{K}|\tanh|\boldsymbol{K}|d \quad (\text{II-43})$$

From (II-42) and (II-43)

$$\beta(\boldsymbol{K})=-\beta(-\boldsymbol{K}) \quad (\text{II-44})$$

Then we indicate the velocity vector of progressive wave as $C^{(+)}$ and that of reverse progressive wave as $C^{(-)}$, and, using the condition of (II-44), we obtain

$$\phi^{(1)}(x,y,z,t)=\int_{-\infty}^{\infty}e^{i(\boldsymbol{K}.\boldsymbol{X}-\text{sgn}(\boldsymbol{K}.\boldsymbol{C}^{(+)})|\beta|t)}d\zeta^{(+)}(\boldsymbol{K})\cosh|\boldsymbol{K}|(z-d)$$

$$+\int_{-\infty}^{\infty}e^{i(\boldsymbol{K}.\boldsymbol{X}-\text{sgn}(\boldsymbol{K}.\boldsymbol{C}^{(-)})|\beta|t)}d\zeta^{(-)}(\boldsymbol{K})\cosh|\boldsymbol{K}|(z-d) \quad (\text{II-45})$$

(II-45) is the general expression of $\phi^{(1)}$, and, because we limit the problem to progressive waves, we use the first integral of (II-45). Omitting the (+) notation, we find $\eta^{(1)}$ by (II-35).

$$\eta^{(1)}=\int_{-\infty}^{\infty}i\,|\boldsymbol{K}|\frac{e^{i(\boldsymbol{K}.\boldsymbol{X}-\text{sgn}(\boldsymbol{K}.\boldsymbol{C})|\beta|t)}}{-\text{sgn}(\boldsymbol{K}.\boldsymbol{C})\,|\beta|}\sinh|\boldsymbol{K}|d\cdot d\zeta(\boldsymbol{K}) \quad (\text{II-46})$$

Putting $\dfrac{i|\boldsymbol{K}|}{-\text{sgn}(\boldsymbol{K}.\boldsymbol{C})\,|\beta|}\sinh|\boldsymbol{K}|d\cdot d\zeta(\boldsymbol{K})=d\zeta_1(\boldsymbol{K})$

$$\eta^{(1)}=\int_{-\infty}^{\infty}e^{i(\boldsymbol{K}.\boldsymbol{X}-\text{sgn}(\boldsymbol{K}.\boldsymbol{C})|\beta|t)}d\zeta_1(\boldsymbol{K}) \quad (\text{II-47})$$

$$\phi^{(1)}(x,y,z,t)=\int_{-\infty}^{\infty}\frac{i\text{sgn}(\boldsymbol{K}.\boldsymbol{C})\,|\beta|}{|\boldsymbol{K}|}\,\frac{e^{i(\boldsymbol{K}.\boldsymbol{X}-\text{sgn}(\boldsymbol{K}.\boldsymbol{C})|\beta|t)}}{\sinh|\boldsymbol{K}|d}\cosh|\boldsymbol{K}|(z-d)\,d\zeta_1(\boldsymbol{K})$$

(II-48)

Inserting $\phi^{(1)}$ of (II-48) into (II-36), we can compute $\phi^{(2)}$.

Const. t in $\phi^{(2)}$ is also neglected in the present case.

$$\phi^{(2)}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Ge^{i\{(\boldsymbol{K}+\boldsymbol{K}').\boldsymbol{X}-(\text{sgn}(\boldsymbol{K}.\boldsymbol{C})|\beta|+\text{sgn}(\boldsymbol{K}'.\boldsymbol{C}')|\beta'|)t\}}$$

$$\times\frac{\cosh|\boldsymbol{K}+\boldsymbol{K}'|(z-d)}{\sinh|\boldsymbol{K}+\boldsymbol{K}'|d}d\zeta_1(\boldsymbol{K})\,d\zeta_1(\boldsymbol{K}') \quad (\text{II-49})$$

Here

$$G=\frac{ig}{2}\;\frac{1}{-(\text{sgn}(\boldsymbol{K}.\boldsymbol{C})|\beta|+\text{sgn}(\boldsymbol{K}'.\boldsymbol{C}')|\beta'|)^2\coth|\boldsymbol{K}+\boldsymbol{K}'|d+g|\boldsymbol{K}+\boldsymbol{K}'|}$$

$$[|\boldsymbol{K}|\text{sgn}(\boldsymbol{K}.\boldsymbol{C})|\beta|\coth|\boldsymbol{K}|d+|\boldsymbol{K}'|\text{sgn}(\boldsymbol{K}'.\boldsymbol{C}')|\beta'|\coth|\boldsymbol{K}'|d$$

$$+2\boldsymbol{K}.\boldsymbol{K}'\frac{\text{sgn}(\boldsymbol{K}.\boldsymbol{C})}{|\boldsymbol{K}|}|\beta|\coth|\boldsymbol{K}|d+2\boldsymbol{K}.\boldsymbol{K}'\frac{\text{sgn}(\boldsymbol{K}'.\boldsymbol{C}')}{|\boldsymbol{K}'|}|\beta'|\coth|\boldsymbol{K}'|d$$

$$-2\text{sgn}(\boldsymbol{K}.\boldsymbol{C})|\beta|\,|\boldsymbol{K}'|\tanh|\boldsymbol{K}'|d-2\text{sgn}(\boldsymbol{K}'.\boldsymbol{C}')|\beta'|\,|\boldsymbol{K}|\tanh|\boldsymbol{K}|d$$

$$-\mathrm{sgn}(\boldsymbol{K}.\boldsymbol{C})|\beta||\boldsymbol{K}|\tanh|\boldsymbol{K}|d-\mathrm{sgn}(\boldsymbol{K}'.\boldsymbol{C}')|\beta'||\boldsymbol{K}'|\tanh|\boldsymbol{K}'|d] \qquad (\text{II-50})$$

Using $\phi^{(2)}$ of (II-49), we obtain $\eta^{(2)}$ by the relation of (II-37).

$$\eta^{(2)}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\frac{1}{2}(\mathrm{sgn}(\boldsymbol{K}.\boldsymbol{C})|\beta|+\mathrm{sgn}(\boldsymbol{K}'.\boldsymbol{C}')|\beta'|)\frac{2}{ig}G$$

$$-\frac{1}{2g}\mathrm{sgn}(\boldsymbol{K}.\boldsymbol{C})|\beta|\mathrm{sgn}(\boldsymbol{K}'.\boldsymbol{C}')|\beta'|+\frac{1}{2g}\frac{\boldsymbol{K}.\boldsymbol{K}'\mathrm{sgn}(\boldsymbol{K}.\boldsymbol{C})|\beta|\mathrm{sgn}(\boldsymbol{K}'.\boldsymbol{C}')|\beta'|}{|\boldsymbol{K}||\boldsymbol{K}'|}$$

$$\times\coth|\boldsymbol{K}|d\coth|\boldsymbol{K}'|d-\frac{1}{2}|\boldsymbol{K}|\tanh|\boldsymbol{K}|d-\frac{1}{2}|\boldsymbol{K}'|\tanh|\boldsymbol{K}'|d\Big]$$

$$\times e^{i[(\boldsymbol{K}+\boldsymbol{K}').\boldsymbol{X}-\{\mathrm{sgn}(\boldsymbol{K}.\boldsymbol{C})|\beta|+\mathrm{sgn}(\boldsymbol{K}'.\boldsymbol{C}')|\beta'|\}t]}d\zeta_1(\boldsymbol{K})d\zeta_1(\boldsymbol{K}') \qquad (\text{II-51})$$

We can put $t=0$ in (II-51), and consider the two-dimensional random process concerned with wave number $\boldsymbol{K}$ and $\boldsymbol{K}'$. Using $H(\boldsymbol{K}, \boldsymbol{K}')$ expressed later,

$$\eta^{(2)}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(\boldsymbol{K},\boldsymbol{K}')e^{i(\boldsymbol{K}+\boldsymbol{K}').\boldsymbol{X}}d\zeta_1(\boldsymbol{K})d\zeta_1(\boldsymbol{K}') \qquad (\text{II-52})$$

We use $\boldsymbol{K}+\boldsymbol{K}'=\boldsymbol{Q}$ (II-53); corresponding to $\omega+\omega'=\lambda$ in one-dimensional case.

$$\eta^{(2)}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(\boldsymbol{K},\boldsymbol{Q}-\boldsymbol{K})e^{i\boldsymbol{Q}.\boldsymbol{X}}d\zeta_1(\boldsymbol{K})d\zeta_1(\boldsymbol{Q}-\boldsymbol{K}) \qquad (\text{II-54})$$

Then the two-dimensional wave profile spectrum of the second order may be shown by

$$\Phi^{(2)}(\boldsymbol{Q})=\int_{-\infty}^{\infty}H^2(\boldsymbol{K},\boldsymbol{Q}-\boldsymbol{K})\Phi^{(1)}(\boldsymbol{K})\Phi^{(1)}(\boldsymbol{Q}-\boldsymbol{K})d\boldsymbol{K} \qquad (\text{II-55})$$

In (II-55) $\Phi^{(1)}(\boldsymbol{K})$ is the wave profile spectrum of the first order, and satisfies the relation

$$<\eta^{(1)2}>=\int_{-\infty}^{\infty}\Phi^{(1)}(\boldsymbol{K})d\boldsymbol{K} \qquad (\text{II-56})$$

Detailed expression of $H(\boldsymbol{K},\boldsymbol{Q}-\boldsymbol{K})$ is

$$H(\boldsymbol{K},\boldsymbol{Q}-\boldsymbol{K})=\frac{1}{2}\frac{\{\mathrm{sgn}(\boldsymbol{K}.\boldsymbol{C})|\beta|+\mathrm{sgn}((\boldsymbol{Q}-\boldsymbol{K}).\boldsymbol{C}')\underline{|\beta(\boldsymbol{Q}-\boldsymbol{K})|}\}}{-\{\mathrm{sgn}(\boldsymbol{K}.\boldsymbol{C})|\beta|+\mathrm{sgn}((\boldsymbol{Q}-\boldsymbol{K}).\boldsymbol{C}')\underline{|\beta(\boldsymbol{Q}-\boldsymbol{K})|}\}^2+g|\boldsymbol{Q}|\tanh|\boldsymbol{Q}|d}$$

$$\times[|\boldsymbol{K}|\mathrm{sgn}(\boldsymbol{K}.\boldsymbol{C})|\beta|\coth|\boldsymbol{K}|d+|\boldsymbol{Q}-\boldsymbol{K}|\mathrm{sgn}((\boldsymbol{Q}-\boldsymbol{K}).\boldsymbol{C}')\underline{|\beta(\boldsymbol{Q}-\boldsymbol{K})|}\coth|\boldsymbol{Q}-\boldsymbol{K}|d$$

$$+2\boldsymbol{K}.(\boldsymbol{Q}-\boldsymbol{K})\frac{\mathrm{sgn}(\boldsymbol{K}.\boldsymbol{C})|\beta|\coth|\boldsymbol{K}|d}{|\boldsymbol{K}|}+2\boldsymbol{K}.(\boldsymbol{Q}-\boldsymbol{K})\frac{\mathrm{sgn}((\boldsymbol{Q}-\boldsymbol{K}).\boldsymbol{C}')}{|\boldsymbol{Q}-\boldsymbol{K}|}$$

$$\times\underline{|\beta(\boldsymbol{Q}-\boldsymbol{K})|}\coth|\boldsymbol{Q}-\boldsymbol{K}|d-2\mathrm{sgn}(\boldsymbol{K}.\boldsymbol{C})|\beta||\boldsymbol{Q}-\boldsymbol{K}|\tanh|\boldsymbol{Q}-\boldsymbol{K}|d$$

$$-2\mathrm{sgn}((\boldsymbol{Q}-\boldsymbol{K}).\boldsymbol{C}')\underline{|\beta(\boldsymbol{Q}-\boldsymbol{K})|}|\boldsymbol{K}|\tanh|\boldsymbol{K}|d-\mathrm{sgn}(\boldsymbol{K}.\boldsymbol{C})|\beta||\boldsymbol{K}|\tanh|\boldsymbol{K}|d$$

$$-\mathrm{sgn}((\boldsymbol{Q}-\boldsymbol{K}).\boldsymbol{C}')\underline{|\beta(\boldsymbol{Q}-\boldsymbol{K})|}|\boldsymbol{Q}-\boldsymbol{K}|\tanh|\boldsymbol{Q}-\boldsymbol{K}|d]$$

$$-\frac{1}{2g}\mathrm{sgn}(\boldsymbol{K}.\boldsymbol{C})|\beta|\mathrm{sgn}((\boldsymbol{Q}-\boldsymbol{K}).\boldsymbol{C}')\underline{|\beta(\boldsymbol{Q}-\boldsymbol{K})|}$$

$$+\frac{1}{2g}\frac{\boldsymbol{K}.(\boldsymbol{Q}-\boldsymbol{K})\mathrm{sgn}(\boldsymbol{K}.\boldsymbol{C})|\beta|\mathrm{sgn}((\boldsymbol{Q}-\boldsymbol{K}).\boldsymbol{C}')\underline{|\beta(\boldsymbol{Q}-\boldsymbol{K})|}\coth|\boldsymbol{K}|d\coth|\boldsymbol{Q}-\boldsymbol{K}|d}{|\boldsymbol{K}||\boldsymbol{Q}-\boldsymbol{K}|}$$

$$-\frac{1}{2}|\boldsymbol{K}|\tanh|\boldsymbol{K}|d-\frac{1}{2}|\boldsymbol{Q}-\boldsymbol{K}|\tanh|\boldsymbol{Q}-\boldsymbol{K}|d \qquad (\text{II-57})$$

II-5 examples of two-dimensional case

In II-4 we obtained the two-dimensional wave profile spectrum of the second order. In this paragraph, by making use of the result of II-4, we make some numerical computation to estimate how the influence is refered to the second order when the first order wave has the angular spreading.

The first treatment is the case of the first order wave which has the following properties; at the first order spectrum (i) $|\boldsymbol{K}|$ is constant, and (ii) from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ the intensity is not varied, and it has not the reverse progressive component. This model indicates the case of an extreme angular spreading. In the Fourier-Stieltjes type expression, the spectrum intensity of the first order wave spreads uniformly on the circle of radius $|\boldsymbol{K}|$. That is

$$\Phi^{(1)}(\boldsymbol{K})=\mathrm{A}\ \mathrm{Dirac}(|\boldsymbol{K}|-k_1) \qquad (\text{II-58})$$

$\langle\eta^{(1)2}\rangle$ concerned to this $\Phi^{(1)}(\boldsymbol{K})$ is

$$\langle\eta^{(1)2}\rangle=\int_0^{\infty}\int_{-\pi}^{\pi}\mathrm{A}\ \mathrm{Dirac}(|\boldsymbol{K}|-k_1)kdkd\theta'$$

$$=\mathrm{A}\,k_1\,2\pi \qquad (\text{II-59})$$

We compute then the second order wave number spectrum $\Phi^{(2)}(\boldsymbol{Q})$, and express $\boldsymbol{Q}$ by $\boldsymbol{Q}=\{|\boldsymbol{Q}|,\ \theta\}$. After some computations, next three expressions are obtained by the relation with angle δ, which is given by the definition $\cos\delta=\frac{|\boldsymbol{Q}|}{2|\boldsymbol{K}|}$, $\frac{\pi}{2}\geqq\delta\geqq0$.

(i) $0\leq|\theta|\leq\frac{\pi}{2}-\delta$

$$\Phi^{(2)}(|\boldsymbol{Q}|,\theta)=4H_{(1)}{}^2(\boldsymbol{K},\boldsymbol{Q}-\boldsymbol{K})\frac{A^2k_1{}^2}{\sqrt{4k_1{}^2-|\boldsymbol{Q}|^2}|\boldsymbol{Q}|} \qquad (\text{II-60})$$

$|\boldsymbol{K}|=|\boldsymbol{Q}-\boldsymbol{K}|=k_1$

$$H_{(1)}(\boldsymbol{K},\boldsymbol{Q}-\boldsymbol{K})=|\beta|\frac{2k_1|\beta|\coth k_1d+(2|\boldsymbol{Q}|^2-4k_1{}^2)\frac{|\beta|}{k_1}\coth k_1d-6|\beta|k_1\tanh k_1d}{-4|\beta|^2+g|\boldsymbol{Q}|\tanh|\boldsymbol{Q}|d}$$

$$-\frac{1}{2g}|\beta|^2+\frac{1}{2g}\frac{\left(\frac{|\boldsymbol{Q}|^2}{2}-k_1{}^2\right)|\beta|^2\coth^2k_1d}{k_1{}^2}-k_1\tanh k_1d \qquad (\text{II-61})$$

(ii) $\frac{\pi}{2}-\delta\leq|\theta|\leq\frac{\pi}{2}+\delta$

$$\Phi^{(2)}(|\boldsymbol{Q}|,\theta)=4H_{(2)}{}^2(\boldsymbol{K},\boldsymbol{Q}-\boldsymbol{K})\frac{A^2k_1{}^2}{\sqrt{4k_1{}^2-|\boldsymbol{Q}|^2}|\boldsymbol{Q}|} \qquad (\text{II-62})$$

$$H_{(2)}(\boldsymbol{K},\boldsymbol{Q}-\boldsymbol{K})=\frac{1}{2g}|\beta|^2-\frac{1}{2g}\frac{\left(\frac{|\boldsymbol{Q}|^2}{2}-k_1{}^2\right)|\beta|^2\coth^2k_1d}{k_1{}^2}-k_1\tanh k_1\,d \qquad (\text{II-63})$$

(iii) $\frac{\pi}{2}+\delta \leq |\theta| \leq \pi$

$$\Phi^{(2)}(|\boldsymbol{Q}|,\theta)=4H_{(1)}{}^2(\boldsymbol{K},\boldsymbol{Q}-\boldsymbol{K})\frac{A^2k_1{}^2}{\sqrt{4k_1{}^2-|\boldsymbol{Q}|^2}|\boldsymbol{Q}|} \qquad (\text{II}-64)$$

$H_{(1)}$ is given by (II-61).

From these expressions, we obtain easily

$$<\eta^{(2)2}>=\int \Phi^{(2)}(\boldsymbol{Q})d\boldsymbol{Q}=8\int_0^{2k_1}\frac{A^2k_1{}^2d|\boldsymbol{Q}|}{\sqrt{4k_1{}^2-|\boldsymbol{Q}|^2}}\Big[H_{(1)}{}^2(\boldsymbol{K},\boldsymbol{Q}-\boldsymbol{K})\Big(\pi-2\cos^{-1}\frac{|\boldsymbol{Q}|}{2k_1}\Big)$$

$$+H_{(2)}{}^2(\boldsymbol{K},\boldsymbol{Q}-\boldsymbol{K})2\cos^{-1}\frac{|\boldsymbol{Q}|}{2k_1}\Big] \qquad (\text{II}-65)$$

The two-dimensional wave number spectrum of the secod order is expressed by (II-60),(II-62) and (II-64) in this case, and in the reference of (II-51) we can see that the first term of the right hand side integral of (II-65) just means the power of frequency spectrum at the twofold frequency of the first order power, and that the second term just appears at the zero frequency.

The second treatment is the case when the power of the two-dimensional wave number spectrum of the first order concentrates to a point. In Fourier-Stieltjes expression

$$\Phi^{(1)}(\boldsymbol{K})=\text{B Dirac}(|\boldsymbol{K}|-k_1)\ \text{Dirac}(\theta')+\text{B Dirac}(|\boldsymbol{K}|-k_1)\text{Dirac}(\theta'-\pi) \qquad (\text{II}-66)$$

, and $<\eta^{(1)2}>$ is

$$<\eta^{(1)2}>=2\text{B}k_1 \qquad (\text{II}-67)$$

If this $<\eta^{(1)2}>$ equals to $<\eta^{(1)2}>$ of (II-59),

$$A=\frac{B}{\pi} \qquad (\text{II}-68)$$

$\Phi^{(2)}(\boldsymbol{Q})$ in this case can be obtained by the computation of $H(\boldsymbol{K},\boldsymbol{Q}-\boldsymbol{K})$ and by the integration concerned in $\boldsymbol{K}$. $<\eta^{(2)2}>$ is expressed by

$$<\eta^{(2)2}>=\int_{-\infty}^{\infty}\Phi^{(2)}(\boldsymbol{Q})d\boldsymbol{Q}.$$

The result of the computation is

$$<\eta^{(2)2}>=2\pi^2A^2k_1{}^4\Big\{\coth^2 k_1d\cdot\Big(1+\frac{3}{2}\ \frac{1}{\sinh^2 k_1d}\Big)^2+\frac{1}{\sinh^2 2k_1d}\Big\} \qquad (\text{II}-69)$$

The first term of the right hand side of (II-69) shows the power of frequency spectrum at the twofold frequency, and the second term means the power at zero frequency. This term at zero frequency relates to the disregard of const. t of $\phi^{(2)}$ in (II-49).

We use here the numerical example of Fig-II-1 for the estimation of the distribution of $<\eta^{(2)2}>$ in (II-65) and (II-69), and assume that the first order power all concentrates to the peak of the spectrum in the frequency spectrum. So in the present notations,

$<\eta^{(1)2}>=2.4586cm^2$, $f_1=1.65c/s$, $k_1=0.1165cm^{-1}$, $A=3.3589cm^3$

Using these numerical values, $<\eta^{(2)2}>$ of (II-65) and of (II-69) are shown numerically.

$$<\eta^{(2)2}>_{(II-65)}=0.0284+0.00657\ cm^2 \qquad (II-70)$$

$$<\eta^{(2)2}>_{(II-69)}=0.0659+0.000151\ cm^2 \qquad (II-71)$$

In both expressions, the first term in the right hand side appears at the twofold frequency, and the second term situates at the zero frequency. From these expressions we see that, in the frequency spectrum, the angular spreading of the first order wave decreases the second order component at the twofold frequency and increases the value at the zero frequency. Though these examples are very simple defining $|K|$ to a single value, they show that the second order nonlinear component at the high frequency part of the frequency spectrum is moderately controlled by the angular spreading of waves.

In Fig-II-2 the estimated value of $<\eta^{(2)2}>$, which obtained in experiment near the twofold frequency of the spectrum peak, is about $0.079cm^2$, and is greater than the value of (II-71). This may be probably caused by the mechanical effect of the analogue analyser.

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