2020/4/6

球座標系における Navier の式

野津

1. はじめに

弾性波動論の支配方程式である Navier の式は直交座標系では次式で与えられる．

$$\rho\ddot{u}_i = \mu u_{i,jj} + (\lambda + \mu)u_{j,ji} + \rho b_i \tag{1}$$

ここにρは密度，λとμはラメ定数，u_iは変位，b_iは物体力である．ここでは，震源から球面状に広がる波動の理解など球座標系（図-1）における Navier の式を求めることを考える．

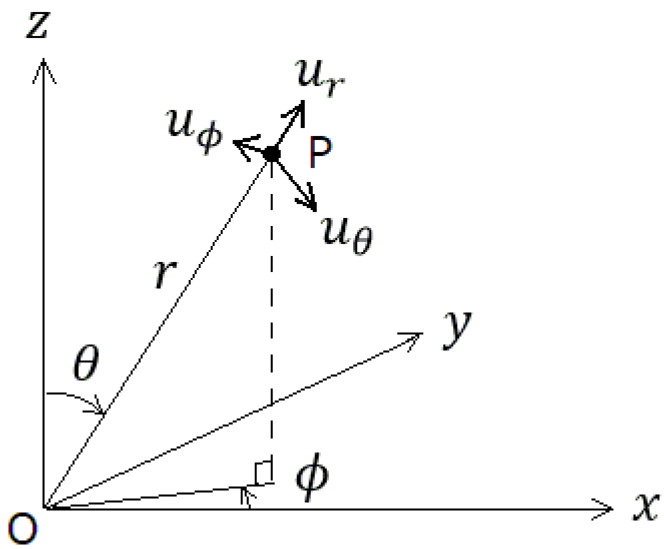

図-1　球座標系（任意の点 P の座標を(r, θ, ϕ)の組み合わせで表示）

2. 球座標系における発散とラプラシアン

球座標系では，図-1 に示すように，任意の点 P の座標を(r, θ, ϕ)の組み合わせで表示する．
直交座標系と球座標系における座標の関係は次式で与えられる．

$$x = r\sin\theta\cos\phi \tag{2}$$
$$y = r\sin\theta\sin\phi \tag{3}$$
$$z = r\cos\theta \tag{4}$$

また，直交座標系と球座標系におけるベクトル場$\boldsymbol{u}$の成分の関係は次式で与えられる．

$$u_x = u_r\sin\theta\cos\phi + u_\theta\cos\theta\cos\phi - u_\phi\sin\phi \tag{5}$$
$$u_y = u_r\sin\theta\sin\phi + u_\theta\cos\theta\sin\phi + u_\phi\cos\phi \tag{6}$$
$$u_z = u_r\cos\theta - u_\theta\sin\theta \tag{7}$$

ここで，球座標系における Navier の式を求めるための準備として，ベクトル場の発散とスカラー場のラプラシアンについて考える．

直交座標系ではベクトル場$\boldsymbol{u}$の発散は次式で与えられる．

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}=\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z} \tag{8}$$

これを球座標系に変換するとき，r方向，θ方向，ϕ方向の空間微分を計算して和をとればよいのであるが，単純に(u_r,u_θ,u_ϕ)の空間微分を計算してしまうと誤りとなる．実際には，ある点 P(r_0,θ_0,ϕ_0)において発散を計算しようとするとき，**図-2** に示すように角度の固定された直交座標系(ξ,η,ζ)をとり，(u_ξ,u_η,u_ζ)の空間微分を計算する必要がある．(u_ξ,u_η,u_ζ)と(u_r,u_θ,u_ϕ)は点 P では一致するが点 P の近傍では一致しないため空間微分の計算結果は異なってくる．

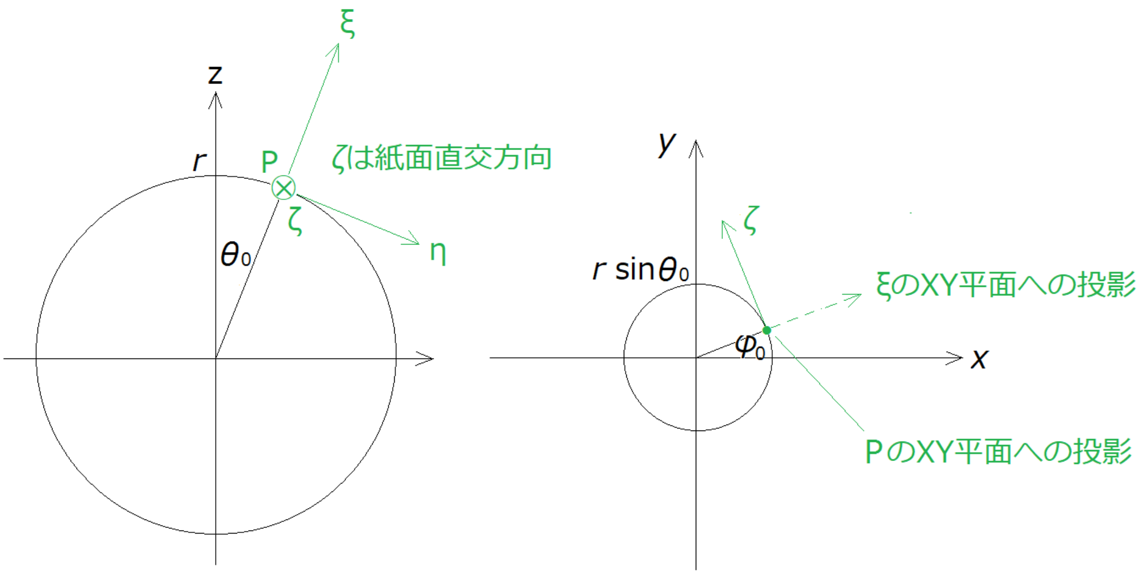

図-2　P を通りz軸を含む平面（左）と$x-y$平面（右）

r方向，θ方向，ϕ方向の単位ベクトルの成分は次式で与えられる．

$$\boldsymbol{e_r}=\begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix},\boldsymbol{e_\theta}=\begin{pmatrix}\cos\theta\cos\phi\\ \cos\theta\sin\phi\\ -\sin\theta\end{pmatrix},\boldsymbol{e_\phi}=\begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix}$$

ξ方向，η方向，ζ方向の単位ベクトルの成分は次式で与えられる．

$$\boldsymbol{e_\xi}=\begin{pmatrix}\sin\theta_0\cos\phi_0\\ \sin\theta_0\sin\phi_0\\ \cos\theta_0\end{pmatrix},\boldsymbol{e_\eta}=\begin{pmatrix}\cos\theta_0\cos\phi_0\\ \cos\theta_0\sin\phi_0\\ -\sin\theta_0\end{pmatrix},\boldsymbol{e_\zeta}=\begin{pmatrix}-\sin\phi_0\\ \cos\phi_0\\ 0\end{pmatrix}$$

これらの内積より，(u_ξ,u_η,u_ζ)と(u_r,u_θ,u_ϕ)との関係は次式で与えられる．

$$
\begin{aligned}
u_\xi = &\{\sin\theta\sin\theta_0\cos(\phi-\phi_0)+\cos\theta\cos\theta_0\}u_r \\
&+\{\cos\theta\sin\theta_0\cos(\phi-\phi_0)-\sin\theta\cos\theta_0\}u_\theta \\
&-\sin\theta_0\sin(\phi-\phi_0)\,u_\phi
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
u_\eta = &\{\sin\theta\cos\theta_0\cos(\phi-\phi_0)-\cos\theta\sin\theta_0\}u_r \\
&+\{\cos\theta\cos\theta_0\cos(\phi-\phi_0)+\sin\theta\sin\theta_0\}u_\theta \\
&-\cos\theta_0\sin(\phi-\phi_0)\,u_\phi
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
u_\zeta = &\sin\theta\sin(\phi-\phi_0)\,u_r \\
&+\cos\theta\sin(\phi-\phi_0)\,u_\theta \\
&+\cos(\phi-\phi_0)\,u_\phi
\end{aligned}
\tag{11}
$$

これを用いると，球座標系における発散は

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{u} &= \frac{\partial u_\xi}{\partial r}+\frac{1}{r}\frac{\partial u_\eta}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial u_\zeta}{\partial\phi} \\
&= \frac{\partial}{\partial r}\{(\sin\theta\sin\theta_0\cos(\phi-\phi_0)+\cos\theta\cos\theta_0)u_r\}+\frac{\partial}{\partial r}\{(\cos\theta\sin\theta_0\cos(\phi-\phi_0)-\sin\theta\cos\theta_0)u_\theta\} \\
&\quad -\frac{\partial}{\partial r}\{\sin\theta_0\sin(\phi-\phi_0)\,u_\phi\} \\
&\quad +\frac{1}{r}\frac{\partial}{\partial\theta}\{(\sin\theta\cos\theta_0\cos(\phi-\phi_0)-\cos\theta\sin\theta_0)u_r\}+\frac{1}{r}\frac{\partial}{\partial\theta}\{(\cos\theta\cos\theta_0\cos(\phi-\phi_0)+\sin\theta\sin\theta_0)u_\theta\} \\
&\quad -\frac{1}{r}\frac{\partial}{\partial\theta}\{\cos\theta_0\sin(\phi-\phi_0)\,u_\phi\} \\
&\quad +\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\{\sin\theta\sin(\phi-\phi_0)\,u_r\}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\{\cos\theta\sin(\phi-\phi_0)\,u_\theta\} \\
&\quad +\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\{\cos(\phi-\phi_0)\,u_\phi\} \\
&= (\sin\theta\sin\theta_0\cos(\phi-\phi_0)+\cos\theta\cos\theta_0)\frac{\partial u_r}{\partial r}+(\cos\theta\sin\theta_0\cos(\phi-\phi_0)-\sin\theta\cos\theta_0)\frac{\partial u_\theta}{\partial r} \\
&\quad -\sin\theta_0\sin(\phi-\phi_0)\frac{\partial u_\phi}{\partial r} \\
&\quad +(\sin\theta\cos\theta_0\cos(\phi-\phi_0)-\cos\theta\sin\theta_0)\frac{1}{r}\frac{\partial u_r}{\partial\theta}+(\cos\theta\cos\theta_0\cos(\phi-\phi_0)+\sin\theta\sin\theta_0)\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} \\
&\quad -\cos\theta_0\sin(\phi-\phi_0)\frac{1}{r}\frac{\partial u_\phi}{\partial\theta} \\
&\quad +(\cos\theta\cos\theta_0\cos(\phi-\phi_0)+\sin\theta\sin\theta_0)\frac{1}{r}u_r+(-\sin\theta\cos\theta_0\cos(\phi-\phi_0)+\cos\theta\sin\theta_0)\frac{1}{r}u_\theta \\
&\quad +\sin\theta\sin(\phi-\phi_0)\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi}+\cos\theta\sin(\phi-\phi_0)\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} \\
&\quad +\cos(\phi-\phi_0)\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} \\
&\quad +\sin\theta\cos(\phi-\phi_0)\frac{1}{r\sin\theta}u_r+\cos\theta\cos(\phi-\phi_0)\frac{1}{r\sin\theta}u_\theta
\end{aligned}
$$

$$-\sin(\phi-\phi_0)\frac{1}{r\sin\theta}u_\phi \tag{12}$$

となる．ここで$\theta\to\theta_0$，$\phi\to\phi_0$とし，θ_0を改めてθと書くことにすれば，球座標系における発散として

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}=\frac{\partial u_r}{\partial r}+\frac{2}{r}u_r+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}u_\theta+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} \tag{13}$$

が得られる．

式(13)において$\boldsymbol{u}=\nabla f$であるような特別な場合を考え（fはスカラー場）

$$u_r=\frac{\partial f}{\partial r},u_\theta=\frac{1}{r}\frac{\partial f}{\partial\theta},u_\phi=\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi} \tag{14}$$

を式(13)に代入すれば，球座標系におけるスカラー場fのラプラシアンとして次式で与えられる．

$$\nabla^2 f=\frac{\partial^2 f}{\partial r^2}+\frac{2}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial f}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} \tag{15}$$

なお，球座標系におけるベクトル場の回転については，以下の Navier の式の検討では用いないが，利用頻度が高いため**付録**に示す（球座標系におけるベクトル場の回転を用いて球座標系における Navier の式を求めることもできる）．

3. 球座標系における Navier の式

まずr方向の式について考える．式(1)は任意の方向について成立しているので，r方向について考えると，左辺が$\rho\ddot{u}_r$，右辺第 3 項がρb_rとなることは自明であり，また，右辺第 2 項は$(\lambda+\mu)\,\partial(\boldsymbol{\nabla}\cdot\boldsymbol{u})/\partial r$となる．しかしながら，右辺第 1 項を$u_r$のラプラシアン（に$\mu$を乗じたもの）としてしまうのは誤りであり，$u_\xi$のラプラシアンを計算する必要がある．

点 $\mathrm{P}(r_0,\theta_0\,,\phi_0)$において$u_\xi$のラプラシアンを計算すると式(9)より

$$\begin{aligned}
\nabla^2 u_\xi &= \left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)u_\xi \\
&= \left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\{(\sin\theta\sin\theta_0\cos(\phi-\phi_0)+\cos\theta\cos\theta_0)u_r\} \\
&\quad +\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\{(\cos\theta\sin\theta_0\cos(\phi-\phi_0)-\sin\theta\cos\theta_0)u_\theta\} \\
&\quad -\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\{\sin\theta_0\sin(\phi-\phi_0)\,u_\phi\} \\
&= (\sin\theta\sin\theta_0\cos(\phi-\phi_0)+\cos\theta\cos\theta_0)\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{2}{r}\frac{\partial u_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_r}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_r}{\partial\phi^2}\right) \\
&\quad +(\cos\theta\sin\theta_0\cos(\phi-\phi_0)-\sin\theta\cos\theta_0)\frac{2}{r^2}\frac{\partial u_r}{\partial\theta} \\
&\quad -(\sin\theta\sin\theta_0\cos(\phi-\phi_0)+\cos\theta\cos\theta_0)\frac{1}{r^2}u_r
\end{aligned}$$

$$+(\cos\theta\sin\theta_0\cos(\phi-\phi_0)-\sin\theta\cos\theta_0)\frac{\cos\theta}{r^2\sin\theta}u_r$$

$$+(-\sin\theta\sin\theta_0\sin(\phi-\phi_0))\frac{2}{r^2\sin^2\theta}\frac{\partial u_r}{\partial\phi}$$

$$+(-\sin\theta\sin\theta_0\cos(\phi-\phi_0))\frac{1}{r^2\sin^2\theta}u_r$$

$$+(\cos\theta\sin\theta_0\cos(\phi-\phi_0)-\sin\theta\cos\theta_0)\left(\frac{\partial^2 u_\theta}{\partial r^2}+\frac{2}{r}\frac{\partial u_\theta}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_\theta}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\theta}{\partial\phi^2}\right)$$

$$-(\sin\theta\sin\theta_0\cos(\phi-\phi_0)+\cos\theta\cos\theta_0)\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}$$

$$-(\cos\theta\sin\theta_0\cos(\phi-\phi_0)-\sin\theta\cos\theta_0)\frac{1}{r^2}u_\theta$$

$$-(\sin\theta\sin\theta_0\cos(\phi-\phi_0)+\cos\theta\cos\theta_0)\frac{\cos\theta}{r^2\sin\theta}u_\theta$$

$$+(-\cos\theta\sin\theta_0\sin(\phi-\phi_0))\frac{2}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}$$

$$+(-\cos\theta\sin\theta_0\cos(\phi-\phi_0))\frac{1}{r^2\sin^2\theta}u_\theta$$

$$-\sin\theta_0\sin(\phi-\phi_0)\left(\frac{\partial^2 u_\phi}{\partial r^2}+\frac{2}{r}\frac{\partial u_\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\phi}{\partial\phi^2}\right)$$

$$-\sin\theta_0\cos(\phi-\phi_0)\frac{2}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}$$

$$+\sin\theta_0\sin(\phi-\phi_0)\frac{1}{r^2\sin^2\theta}u_\phi$$

となる．ここで$\theta\to\theta_0$，$\phi\to\phi_0$とし，θ_0を改めてθと書けば,

$$\nabla^2 u_\xi=\frac{\partial^2 u_r}{\partial r^2}+\frac{2}{r}\frac{\partial u_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_r}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_r}{\partial\phi^2}$$
$$-\frac{2}{r^2}u_r-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{2\cos\theta}{r^2\sin\theta}u_\theta-\frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}$$

よって，r方向の式は

$$\rho\ddot{u}_r=\mu\left[\frac{\partial^2 u_r}{\partial r^2}+\frac{2}{r}\frac{\partial u_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_r}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_r}{\partial\phi^2}-\frac{2}{r^2}u_r-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{2\cos\theta}{r^2\sin\theta}u_\theta-\frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right]$$
$$+(\lambda+\mu)\frac{\partial}{\partial r}\boldsymbol{\nabla}\cdot\boldsymbol{u}+\rho b_r$$

となる．次にθ方向の式を考えるため点 $\mathrm{P}(r_0,\theta_0,\phi_0)$において$u_\eta$のラプラシアンを計算すると式(10)より

$$\nabla^2 u_\eta=\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)u_\eta$$

$$= \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\{(\sin\theta\cos\theta_0\cos(\phi-\phi_0) - \cos\theta\sin\theta_0)u_r\}$$

$$+ \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\{(\cos\theta\cos\theta_0\cos(\phi-\phi_0) + \sin\theta\sin\theta_0)u_\theta\}$$

$$- \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\{\cos\theta_0\sin(\phi-\phi_0)\,u_\phi\}$$

$$= (\sin\theta\cos\theta_0\cos(\phi-\phi_0) - \cos\theta\sin\theta_0)\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{2}{r}\frac{\partial u_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_r}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_r}{\partial\phi^2}\right)$$

$$+(\cos\theta\cos\theta_0\cos(\phi-\phi_0) + \sin\theta\sin\theta_0)\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}$$

$$+(-\sin\theta\cos\theta_0\cos(\phi-\phi_0) + \cos\theta\sin\theta_0)\frac{1}{r^2}u_r$$

$$+(\cos\theta\cos\theta_0\cos(\phi-\phi_0) + \sin\theta\sin\theta_0)\frac{\cos\theta}{r^2\sin\theta}u_r$$

$$-(\sin\theta\cos\theta_0\sin(\phi-\phi_0))\frac{2}{r^2\sin^2\theta}\frac{\partial u_r}{\partial\phi}$$

$$-(\sin\theta\cos\theta_0\cos(\phi-\phi_0))\frac{1}{r^2\sin^2\theta}u_r$$

$$+(\cos\theta\cos\theta_0\cos(\phi-\phi_0) + \sin\theta\sin\theta_0)\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{2}{r}\frac{\partial u_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_\theta}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\theta}{\partial\phi^2}\right)$$

$$+(-\sin\theta\cos\theta_0\cos(\phi-\phi_0) + \cos\theta\sin\theta_0)\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}$$

$$-(\cos\theta\cos\theta_0\cos(\phi-\phi_0) + \sin\theta\sin\theta_0)\frac{1}{r^2}u_\theta$$

$$+(-\sin\theta\cos\theta_0\cos(\phi-\phi_0) + \cos\theta\sin\theta_0)\frac{\cos\theta}{r^2\sin\theta}u_\theta$$

$$+(-\cos\theta\cos\theta_0\sin(\phi-\phi_0))\frac{2}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}$$

$$+(-\cos\theta\cos\theta_0\cos(\phi-\phi_0))\frac{1}{r^2\sin^2\theta}u_\theta$$

$$-\cos\theta_0\sin(\phi-\phi_0)\left(\frac{\partial^2 u_\phi}{\partial r^2} + \frac{2}{r}\frac{\partial u_\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\phi}{\partial\phi^2}\right)$$

$$-\cos\theta_0\cos(\phi-\phi_0)\frac{2}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}$$

$$+\cos\theta_0\sin(\phi-\phi_0)\frac{1}{r^2\sin^2\theta}u_\phi$$

となる．ここで$\theta \to \theta_0$，$\phi \to \phi_0$とし，θ_0を改めてθと書けば，

$$\nabla^2 u_\eta = \frac{\partial^2 u_\theta}{\partial r^2} + \frac{2}{r}\frac{\partial u_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_\theta}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\theta}{\partial\phi^2}$$
$$+ \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{1}{r^2\sin^2\theta}u_\theta - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}$$

よって，θ方向の式は

$$\rho\ddot{u}_\theta = \mu\left[\frac{\partial^2 u_\theta}{\partial r^2} + \frac{2}{r}\frac{\partial u_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\theta}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{1}{r^2\sin^2\theta}u_\theta - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial \phi}\right]$$

$$+(\lambda+\mu)\frac{1}{r}\frac{\partial}{\partial\theta}\boldsymbol{\nabla}\cdot\boldsymbol{u} + \rho b_\theta$$

となる．さらにϕ方向の式を考えるため点 P$(r_0, \theta_0\ , \phi_0)$において$u_\zeta$のラプラシアンを計算すると式(11)より

$$\nabla^2 u_\zeta = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)u_\zeta$$

$$= \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\{\sin\theta\sin(\phi-\phi_0)\,u_r\}$$

$$+\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\{\cos\theta\sin(\phi-\phi_0)\,u_\theta\}$$

$$+\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\{\cos(\phi-\phi_0)\,u_\phi\}$$

$$= \sin\theta\sin(\phi-\phi_0)\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{2}{r}\frac{\partial u_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_r}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_r}{\partial\phi^2}\right)$$

$$+\cos\theta\sin(\phi-\phi_0)\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}$$

$$-\sin\theta\sin(\phi-\phi_0)\frac{1}{r^2}u_r$$

$$+\cos\theta\sin(\phi-\phi_0)\frac{\cos\theta}{r^2\sin\theta}u_r$$

$$+\sin\theta\cos(\phi-\phi_0)\frac{2}{r^2\sin^2\theta}\frac{\partial u_r}{\partial\phi}$$

$$-\sin\theta\sin(\phi-\phi_0)\frac{1}{r^2\sin^2\theta}u_r$$

$$+\cos\theta\sin(\phi-\phi_0)\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{2}{r}\frac{\partial u_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_\theta}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\theta}{\partial\phi^2}\right)$$

$$-\sin\theta\sin(\phi-\phi_0)\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}$$

$$-\cos\theta\sin(\phi-\phi_0)\frac{1}{r^2}u_\theta$$

$$-\sin\theta\sin(\phi-\phi_0)\frac{\cos\theta}{r^2\sin\theta}u_\theta$$

$$+\cos\theta\cos(\phi-\phi_0)\frac{2}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}$$

$$-\cos\theta\sin(\phi-\phi_0)\frac{1}{r^2\sin^2\theta}u_\theta$$

$$+\cos(\phi-\phi_0)\left(\frac{\partial^2 u_\phi}{\partial r^2} + \frac{2}{r}\frac{\partial u_\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\phi}{\partial\phi^2}\right)$$

$$-\sin(\phi-\phi_0)\frac{2}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}$$

$$-\cos(\phi-\phi_0)\frac{1}{r^2\sin^2\theta}u_\phi$$

となる．ここで$\theta\to\theta_0$，$\phi\to\phi_0$とし，θ_0を改めてθと書けば，

$$\nabla^2 u_\zeta=\frac{\partial^2 u_\phi}{\partial r^2}+\frac{2}{r}\frac{\partial u_\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial \theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\phi}{\partial\phi^2}$$
$$+\frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}-\frac{1}{r^2\sin^2\theta}u_\phi$$

よって，ϕ方向の式は

$$\rho\ddot{u}_\phi=\mu\left[\frac{\partial^2 u_\phi}{\partial r^2}+\frac{2}{r}\frac{\partial u_\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial \theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\phi}{\partial\phi^2}+\frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}-\frac{1}{r^2\sin^2\theta}u_\phi\right]$$
$$+(\lambda+\mu)\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\boldsymbol{\nabla}\cdot\boldsymbol{u}+\rho b_\phi$$

これらをまとめると，球座標系における Navier の式は結局次式となる.

$$\rho\ddot{u}_r=\mu\left[\frac{\partial^2 u_r}{\partial r^2}+\frac{2}{r}\frac{\partial u_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_r}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_r}{\partial\phi^2}-\frac{2}{r^2}u_r-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{2\cos\theta}{r^2\sin\theta}u_\theta-\frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right]$$
$$+(\lambda+\mu)\frac{\partial}{\partial r}\boldsymbol{\nabla}\cdot\boldsymbol{u}+\rho b_r \tag{16}$$

$$\rho\ddot{u}_\theta=\mu\left[\frac{\partial^2 u_\theta}{\partial r^2}+\frac{2}{r}\frac{\partial u_\theta}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_\theta}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\theta}{\partial\phi^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{1}{r^2\sin^2\theta}u_\theta-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}\right]$$
$$+(\lambda+\mu)\frac{1}{r}\frac{\partial}{\partial\theta}\boldsymbol{\nabla}\cdot\boldsymbol{u}+\rho b_\theta \tag{17}$$

$$\rho\ddot{u}_\phi=\mu\left[\frac{\partial^2 u_\phi}{\partial r^2}+\frac{2}{r}\frac{\partial u_\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial \theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\phi}{\partial\phi^2}+\frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}-\frac{1}{r^2\sin^2\theta}u_\phi\right]$$
$$+(\lambda+\mu)\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\boldsymbol{\nabla}\cdot\boldsymbol{u}+\rho b_\phi \tag{18}$$

ここで変位場の発散$\boldsymbol{\nabla}\cdot\boldsymbol{u}$は先に見たとおり次式で与えられる.

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}=\frac{\partial u_r}{\partial r}+\frac{2}{r}u_r+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}u_\theta+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} \tag{19}$$

4. 球座標系における応力～変位関係

球座標系における応力～変位関係もよく用いられるのでここで整理しておく. まず, 応力～ひずみ関係は, 直交座標系における応力～ひずみ関係より次式で与えられる.

$$\sigma_{rr}=(\lambda+2\mu)\varepsilon_{rr}+\lambda\varepsilon_{\theta\theta}+\lambda\varepsilon_{\phi\phi} \tag{20}$$
$$\sigma_{\theta\theta}=\lambda\varepsilon_{rr}+(\lambda+2\mu)\varepsilon_{\theta\theta}+\lambda\varepsilon_{\phi\phi} \tag{21}$$
$$\sigma_{\phi\phi}=\lambda\varepsilon_{rr}+\lambda\varepsilon_{\theta\theta}+(\lambda+2\mu)\varepsilon_{\phi\phi} \tag{22}$$

$$\sigma_{r\theta} = 2\mu\varepsilon_{r\theta} \tag{23}$$

$$\sigma_{\theta\phi} = 2\mu\varepsilon_{\theta\phi} \tag{24}$$

$$\sigma_{r\phi} = 2\mu\varepsilon_{r\phi} \tag{25}$$

一方, ひずみ～変位関係については, (u_ξ, u_η, u_ζ)と(u_r, u_θ, u_ϕ)との違いに注意しながら微分し$\theta \to \theta_0, \phi \to \phi_0$とすると

$$\varepsilon_{rr} = \frac{\partial u_\xi}{\partial r} = \frac{\partial u_r}{\partial r} \tag{26}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\eta}{\partial \theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \tag{27}$$

$$\varepsilon_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\zeta}{\partial \phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{\cos\theta}{r\sin\theta}u_\theta \tag{28}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_\xi}{\partial \theta} + \frac{\partial u_\eta}{\partial r}\right) = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right) \tag{29}$$

$$\varepsilon_{\theta\phi} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_\eta}{\partial \phi} + \frac{1}{r}\frac{\partial u_\zeta}{\partial \theta}\right) = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} - \frac{\cos\theta}{r\sin\theta}u_\phi + \frac{1}{r}\frac{\partial u_\phi}{\partial \theta}\right) \tag{30}$$

$$\varepsilon_{r\phi} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_\xi}{\partial \phi} + \frac{\partial u_\zeta}{\partial r}\right) = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r} + \frac{\partial u_\phi}{\partial r}\right) \tag{31}$$

となる．式(26)−(31)を式(20)–(25)に代入すると応力～変位関係として次式が得られる．

$$\sigma_{rr} = (\lambda + 2\mu)\frac{\partial u_r}{\partial r} + \lambda\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) + \lambda\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{\cos\theta}{r\sin\theta}u_\theta\right) \tag{32}$$

$$\sigma_{\theta\theta} = \lambda\frac{\partial u_r}{\partial r} + (\lambda + 2\mu)\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) + \lambda\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{\cos\theta}{r\sin\theta}u_\theta\right) \tag{33}$$

$$\sigma_{\phi\phi} = \lambda\frac{\partial u_r}{\partial r} + \lambda\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) + (\lambda + 2\mu)\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{\cos\theta}{r\sin\theta}u_\theta\right) \tag{34}$$

$$\sigma_{r\theta} = \mu\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right) \tag{35}$$

$$\sigma_{\theta\phi} = \mu\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} - \frac{\cos\theta}{r\sin\theta}u_\phi + \frac{1}{r}\frac{\partial u_\phi}{\partial \theta}\right) \tag{36}$$

$$\sigma_{r\phi} = \mu\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r} + \frac{\partial u_\phi}{\partial r}\right) \tag{37}$$

付録　球座標系におけるベクトル場の回転

直交座標系ではベクトル場$\boldsymbol{u}$の回転は次式で与えられる．

$$\boldsymbol{\nabla}\times\boldsymbol{u}=\begin{pmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{pmatrix}\times\begin{pmatrix}u_x\\ u_y\\ u_z\end{pmatrix} \tag{A1}$$

この式は(x,y,z)以外の直交座標系にも適用できるので，直交座標系(ξ,η,ζ)に適用すると次式が得られる．

$$(\boldsymbol{\nabla}\times\boldsymbol{u})_r=(\boldsymbol{\nabla}\times\boldsymbol{u})_\xi=\frac{\partial u_\zeta}{\partial\eta}-\frac{\partial u_\eta}{\partial\zeta}=\frac{1}{r}\frac{\partial u_\zeta}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial u_\eta}{\partial\phi} \tag{A2}$$

$$(\boldsymbol{\nabla}\times\boldsymbol{u})_\theta=(\boldsymbol{\nabla}\times\boldsymbol{u})_\eta=\frac{\partial u_\xi}{\partial\zeta}-\frac{\partial u_\zeta}{\partial\xi}=\frac{1}{r\sin\theta}\frac{\partial u_\xi}{\partial\phi}-\frac{\partial u_\zeta}{\partial r} \tag{A3}$$

$$(\boldsymbol{\nabla}\times\boldsymbol{u})_\phi=(\boldsymbol{\nabla}\times\boldsymbol{u})_\zeta=\frac{\partial u_\eta}{\partial\xi}-\frac{\partial u_\xi}{\partial\eta}=\frac{\partial u_\eta}{\partial r}-\frac{1}{r}\frac{\partial u_\xi}{\partial\theta} \tag{A4}$$

ここから先，(u_ξ,u_η,u_ζ)と(u_r,u_θ,u_ϕ)との違いに注意しながら微分し$\theta\to\theta_0$，$\phi\to\phi_0$とすると

$$\begin{aligned}(\boldsymbol{\nabla}\times\boldsymbol{u})_r=&\frac{1}{r}\frac{\partial}{\partial\theta}\{\sin\theta\sin(\phi-\phi_0)\,u_r\}\\
&+\frac{1}{r}\frac{\partial}{\partial\theta}\{\cos\theta\sin(\phi-\phi_0)\,u_\theta\}\\
&+\frac{1}{r}\frac{\partial}{\partial\theta}\{\cos(\phi-\phi_0)\,u_\phi\}\\
&-\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\{(\sin\theta\cos\theta_0\cos(\phi-\phi_0)-\cos\theta\sin\theta_0)u_r\}\\
&-\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\{(\cos\theta\cos\theta_0\cos(\phi-\phi_0)+\sin\theta\sin\theta_0)u_\theta\}\\
&+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\{\cos\theta_0\sin(\phi-\phi_0)\,u_\phi\}\\
=&\frac{1}{r}\frac{\partial u_\phi}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}+\frac{\cos\theta}{r\sin\theta}u_\phi\end{aligned}$$

$$\begin{aligned}(\boldsymbol{\nabla}\times\boldsymbol{u})_\theta=&\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\{(\sin\theta\sin\theta_0\cos(\phi-\phi_0)+\cos\theta\cos\theta_0)u_r\}\\
&+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\{(\cos\theta\sin\theta_0\cos(\phi-\phi_0)-\sin\theta\cos\theta_0)u_\theta\}\\
&-\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\{\sin\theta_0\sin(\phi-\phi_0)\,u_\phi\}\\
&-\frac{\partial}{\partial r}\{\sin\theta\sin(\phi-\phi_0)\,u_r\}\\
&-\frac{\partial}{\partial r}\{\cos\theta\sin(\phi-\phi_0)\,u_\theta\}\end{aligned}$$

$$-\frac{\partial}{\partial r}\left\{\cos(\phi-\phi_0)\,u_\phi\right\}$$

$$=\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi}-\frac{u_\phi}{r}-\frac{\partial u_\phi}{\partial r}$$

$$(\boldsymbol{\nabla}\times\boldsymbol{u})_\phi=\frac{\partial}{\partial r}\{(\sin\theta\cos\theta_0\cos(\phi-\phi_0)-\cos\theta\sin\theta_0)u_r\}$$

$$+\frac{\partial}{\partial r}\{(\cos\theta\cos\theta_0\cos(\phi-\phi_0)+\sin\theta\sin\theta_0)u_\theta\}$$

$$-\frac{\partial}{\partial r}\left\{\cos\theta_0\sin(\phi-\phi_0)\,u_\phi\right\}$$

$$-\frac{1}{r}\frac{\partial}{\partial \theta}\{(\sin\theta\sin\theta_0\cos(\phi-\phi_0)+\cos\theta\cos\theta_0)u_r\}$$

$$-\frac{1}{r}\frac{\partial}{\partial \theta}\{(\cos\theta\sin\theta_0\cos(\phi-\phi_0)-\sin\theta\cos\theta_0)u_\theta\}$$

$$+\frac{1}{r}\frac{\partial}{\partial \theta}\left\{\sin\theta_0\sin(\phi-\phi_0)\,u_\phi\right\}$$

$$=\frac{\partial u_\theta}{\partial r}-\frac{1}{r}\frac{\partial u_r}{\partial \theta}+\frac{u_\theta}{r}$$

これらをまとめると，球座標系におけるベクトル場の回転は以下の通りとなる．

$$(\boldsymbol{\nabla}\times\boldsymbol{u})_r=\frac{1}{r}\frac{\partial u_\phi}{\partial \theta}-\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}+\frac{\cos\theta}{r\sin\theta}u_\phi \tag{A5}$$

$$(\boldsymbol{\nabla}\times\boldsymbol{u})_\theta=\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi}-\frac{u_\phi}{r}-\frac{\partial u_\phi}{\partial r} \tag{A6}$$

$$(\boldsymbol{\nabla}\times\boldsymbol{u})_\phi=\frac{\partial u_\theta}{\partial r}-\frac{1}{r}\frac{\partial u_r}{\partial \theta}+\frac{u_\theta}{r} \tag{A7}$$